A TEXT BOOK OF HIGHER SECONDARY

Elective Mathematics

[IN ENGLISH]

For Class X

2660

KESHAB CHANDRA NAG

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Written strictly in accordance with the New Syllabus for Higher Secondary and Multipurpose Schools for Class X.

A TEXT BOOK OF

HIGHER SECONDARY

ELECTIVE MATHEMATICS

(***IN ENGLISH***)

[MENSURATION, ALGEBRA, PLANE & SOLID GEOMETRY, CO-ORDINATE GEOMETRY & TRIGONOMETRY COMBINED]

PART II

(*FOR CLASS X*)

INSTITUTE OF EDUCATION FOR WOMEN
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BY

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INTRODUCTION

The English edition of the first part (**for class IX**) of my most popular book 'Higher Secondary Elective Mathematics' Parts I—III (in Bengali) was published in November last and has been highly appreciated by teachers and students alike. Being thus encouraged I now publish the English edition of the second part of the book under the title "A Text Book of Higher Secondary Elective Mathematics" (in English) Part II for class X.

It is a **complete text book** for class X of Higher Secondary Schools and **covers the entire Syllabi** for Mensuration, Algebra Plane Geometry, Solid Geometry, Trigonometry and Co-ordinate Geometry.

In this part also all the Topics have been lucidly and elaborately dealt with and illustrated with copious examples culled from the Examination papers of different Universities.

A special feature of this book is that it contains the relevant portions of the question papers set in the West Bengal Higher Secondary, C. U. Pre-University and B. U. Entrance Examinations with answers upto date. I hope the students will find it very helpful to them.

I shall be grateful to the teachers if this book is received by them with sympathy and introduced as a text book in their Schools.

Any suggestion for the improvement of the book will be highly appreciated.

August, 1963, }
CALCUTTA

KESHAB CHANDRA NAG
AUTHOR

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BOARD OF SECONDARY EDUCATION, WEST BENGAL
HIGHER SECONDARY COURSE

Mathematics (Elective Subject)

Class X

ALGEBRA :

Elementary ideas of elimination ; A. P. and G. P. (finite series), H. P. (definition only) ; Variations ; Logarithms (Note —Use of slide rule may be encouraged) :

Irrational quantities, Simultaneous equation in two unknowns of which one is quadratic and the other linear.

GEOMETRY :

THEORETICAL

The angles made by a tangent to a circle with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle.

If two chords of a circle intersect either inside or outside the circle, the rectangle contained by the parts of one is equal to the rectangle contained by the parts of the other. (Note—This proposition may be proved with the help of the properties of similar triangles).

PRACTICAL

Construction of tangents to a circle and of common tangents to two circles (both cases), Construction of regular figures of 3, 4, 5 or 6 sides in or about a circle.

Construction of a mean proportional to two given straight lines.

Construction of a square equal in area to a given polygon.

SOLID GEOMETRY :

Axioms (i). One and only one plane may be made to pass through any two intersecting straight lines.

Axioms (ii). Two intersecting planes cut one another in a straight line and in no point outside it.

To prove :

1. If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it is also perpendicular to the plane in which they lie.

2. All straight lines drawn perpendicular to a given straight line at a given point of it are coplanar.

3. If two straight lines are parallel and if one of them is perpendicular to a plane, then the other is also perpendicular to the plane.

Concept of angle between two planes, angle between a straight line and a plane. Concept of parallelism of planes. Concept of a line being parallel to a plane. Concept of skew lines.

CO-ORDINATE GEOMETRY :

Rectangular cartesian co-ordinates in a plane ; Lengths of segments ; Section of a finite segment in a given ratio ; Area of a triangle ; Straight line.

MENSURATION :

Parallelopipeds, Right Circular cones, Prisms and Pyramids (Expressions without proof, of the surfaces and volumes of these solids).

TRIGONOMETRY :

Trigonometrical ratios of any angle ; Trigonometrical ratios of angles associated with a given angle ; Addition and subtraction formulas : Transformation of products and sums ; Multiple and sub-multiple angles.

Note—It is recommended that Solid Geometry and Mensuration of Solids be taught through the drawing board, and the making and handing of Solid models.

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CHAPTER I

MENSURATION

SOLIDS

Definitions :

You know that a book or a brick or a ball is a solid body and each occupies certain space.

A **Solid** has length, breadth and thickness (or depth or height).

So a solid is of three dimensions.

1. Any portion of space bounded by one or many surfaces, plane or curved, is called a solid body or **solid.**

Thus solids are bounded by surfaces.

[**N. B.** A surface has length and breadth, but no thickness. It is of two dimensions. It may be plane or curved.]

2. **Polyhedron** : A polyhedron is a solid bounded by several plane faces.

[**N. B.** Just as at least three st. lines are necessary to bound a portion of a plane, so at least 4 planes (i.e., plane surfaces) are necessary to bound a certain space.]

These surfaces are called the **faces** of the solid and the straight line in which two surfaces intersect or meet is called the **edge.**

Example. A box or a brick is bounded by six plane surfaces and these surfaces intersect in 12 straight lines.

A cricket ball, however, is bounded by a single curved surface.

3. A solid bounded by three pairs of parallel plane surfaces (or faces) is called a **parallelopiped.**

Each of its faces is a parallelogram and each two opposite faces are equal in all respects.

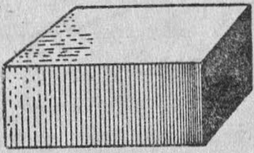


Fig. 1

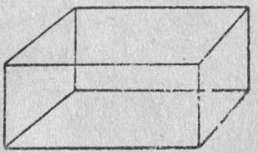


Fig. 2

4. A parallelopiped whose faces are rectangles is called a **rectangular parallelopiped** or a **rectangular solid** or a **cuboid.**

As for instance a wooden box. It has six faces, one at the bottom, one at the top and 4 faces on the four sides.

In fig. 3, the faces are OAQB, CRPS, OASC, OBRC, BRPQ and AQPS.

A rectangular parallelopiped has length, breadth and thickness (or height).

Volume : The volume of a solid is the space contained within its bounding surfaces.

The volume has three dimensions (length, breadth and thickness), while a surface has two dimensions (length and breadth).

Hence a volume is expressed in cubic units and an area of a surface is expressed in square units.

Let the length of a rectangular parallelopiped be a units, the breadth b units and the hight c units of length. Hence

(a) the **area of the surfaces** of a rectangular parallelopiped

$=(2ab+2bc+2ca)$ sq. units

$=\mathbf{2(ab+bc+ca)}$ sq. units.

(b) The **volume** of a rectangular parallelopiped

$=$ **area** of the base $\times$ height

$=$ length $\times$ breadth $\times$ height $=$ **abc** cubic units.

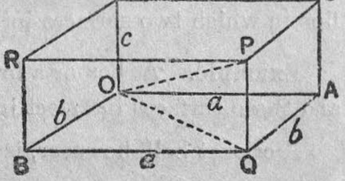


Fig. 3

(c) In figure 3, OP is the diagonal of the rectangular parallelopiped.

$\therefore \quad OP^2 = OQ^2 + PQ^2$ [$\because$ $\angle OQP$ is a rt. angle]

$= OA^2 + AQ^2 + PQ^2$ [$\because$ $\angle OAQ$ is a rt. angle]

$= a^2 + b^2 + c^2.$

$\therefore$ The **diagonal** $= \sqrt{a^2+b^2+c^2}$

$= \sqrt{(\text{length})^2 + (\text{width})^2 + (\text{height})^2}.$

5. If the faces of a rectangular parallelopiped be squares and its length, breadth and height be equal, then it is called a **cube.**

A cube has six equal square surfaces, each opposite pair being parallel. Its angles are all right angles.

If its length, breadth and height each be a units of length, then

(i) the **area of the whole surface of the cube**

$= 2a^2 + 2a^2 + 2a^2 = 6a^2.$

(ii) the **volume of the cube** $= \mathbf{a^3}.$

(iii) the **diagonal** of the cube

$= \sqrt{3a^2} = \mathbf{a}\sqrt{3}.$

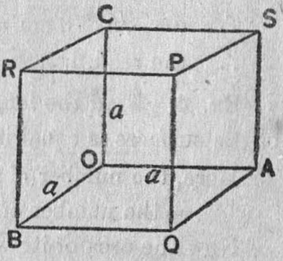


Fig. 4

Examples (1)

Ex. 1. Find the area of the whole surface, the volume and the diagonal of a rectangular solid whose dimensions are 4 yds., 1 yd. 1 ft. and 3 ft.

Here, the length $(a) = 4$ yds. $= 12$ ft.,

the breadth $(b) = 1$ yd. 1 ft. $= 4$ ft. and height $(c) = 3$ ft.

$\therefore$ the reqd. area of the whole surface $= 2(ab+bc+ca)$

$= 2(12 \times 4 + 4 \times 3 + 3 \times 12)$ sq. ft. $= 192$ sq. ft.

Its volume $= abc = 12$ ft. $\times 4$ ft. $\times 3$ ft. $= 144$ cubic ft.

Its diagonal $= \sqrt{a^2+b^2+c^2} = \sqrt{12^2+4^2+3^2}$ ft.

$= \sqrt{169}$ ft. $= 13$ **ft.**

Ex. 2. Find the area of the whole surface, the volume and the diagonal of a cube, each edge of which is 7 feet.

The area of the whole surface of the cube

$=6a^2=6\times7^2$ sq. ft. $=294$ sq. ft.

Its volume $=a^3=(7)^3$ cu. ft. $=343$ cu. ft.

Its diagonal $=a\sqrt{3}=7\times\sqrt{3}$ ft. $=7\sqrt{3}$ ft.

Ex. 3. What is the length of the edge of a cube of which the total area of the surfaces is 346·56 sq. cm.? [C. U. 1956]

Let each edge of the cube be a cm.

Then the area of its whole surface $=6a^2$ sq. cm.

$$\therefore\quad 6a^2=346{\cdot}56 \text{ sq. cm., or, } a^2=\frac{346{\cdot}56}{6} \text{ sq. cm.}=57{\cdot}76 \text{ sq. cm.,}$$

$\therefore\quad a=\sqrt{57{\cdot}76}$ cm. $=7{\cdot}6$ cm.

$\therefore$ the reqd. length of the edge $=7{\cdot}6$ cm.

Ex. 4. Find the length of the edge of a cube whose total area of the surfaces is equal in magnitude to the volume of the cube.

Here, the number of sq. units in the surface

= the number of cubic units in the volume.

Now the area of its whole surface $=6a^2$ sq. units

and its volume $=a^3$ cubic units.

$\therefore\quad a^3=6a^2,\quad\therefore\quad a=6.$

Hence, the length of its edge = 6 units of length.

Ex. 5. The dimensions of a rectangular solid are as 4 : 3 : 2 and its total surface is 1300 sq. in.; find its length, breadth and height.

Here, the length, breadth and height are in the ratio 4 : 3 : 2. So if the length be $4a$ inches, the breadth and height will be $3a$ and $2a$ inches respectively.

Now, the area of its total surface

$=2(4a\times3a+4a\times2a+3a\times2a)$ sq. in. $=52a^2$ sq. in.

$\therefore\quad 52a^2=1300$ sq. in., or, $a^2=25$ sq. in., $\therefore\quad a=5$ in.

Hence, the reqd. length $=4a=4\times5$ in. $=20$ inches,

the breadth $=5\times3$ in. $=15$ in., and height $=5\times2$ in. $=10$ in.

Ex. 6. Three cubes of metal whose sides are 3, 4 and 5 inches respectively are melted and formed into a single cube, show that the edge of the new cube will be 6 inches. [P. U.]

The total volume of the 3 cubes $=\{(3)^3+(4)^3+(5)^3\}$ cu. in.

$=216$ cu. in.

$\therefore$ the volume of the new cube $=216$ cu. in.

$\therefore$ its $(\text{edge})^3=216$ cu. in.

$\therefore$ the length of its edge

$=\sqrt[3]{216}$ in. $=\sqrt[3]{6\times6\times6}$ in. $=6$ inches.

Ex. 7. Find the length of the longest rod that can be placed in a room 30 ft. long, 24 ft. broad and 18 ft. high. [P. U.]

Here, it is evident that the length of the longest rod = the diagonal of the room.

Now, the diagonal $=\sqrt{a^2+b^2+c^2}$, here $a=30$ ft., $b=24$ ft., $c=18$ ft.

$\therefore$ The reqd. length of the rod $=\sqrt{30^2+24^2+18^2}$ ft.

$=\sqrt{1800}$ ft. $=30\sqrt{2}$ ft.

$=42{\cdot}42$ ft. (App.).

Ex. 8. A closed box which externally measures 16 in. long, 12 in. broad and 8 in. high, is made of wood half an inch thick. Find the cost of painting its inner surface at 1 *a.* 6 *p.* per sq. foot.

As the wood is $\frac{1}{2}$ inch thick, the internal dimensions of the box are each less by $\frac{1}{2}$ in. $\times 2$ or 1 in. than the external dimensions. Hence, internally the length $=15$ in., breadth $=11$ in. and height $=7$ in.

$\therefore$ The whole inner surface $=2(15\times11+11\times7+15\times7)$ sq. in.

$=694$ sq. in. $=\frac{694}{144}$ sq. ft.

$\because$ The cost for 1 sq. ft. $=\frac{3}{2}$ annas,

$\therefore$ The reqd. cost of painting $=\frac{3}{2}\,a.\times\frac{694}{144}=\frac{347}{48}\,a.=7\,a.\ 2\frac{3}{4}\,p.$

[**N. B.** If the box be without any lid, the internal height would be $\frac{1}{2}$ inch less, but the other dimensions would be 1 inch less than the external dimensions.]

Ex. 9. A school room is to be built to accommodate 70 children, so as to allow $8\frac{1}{2}$ sq. ft. at floor and $110\frac{1}{2}$ cubic ft. of space for each child ; if the room be 34 ft. long, what must be its breadth and height. [S. A.]

Each boy requires $8\frac{1}{2}$ sq. ft. of floor,

$\therefore$ the area of the floor of the room $= 8\frac{1}{2}$ sq. ft. $\times 70$

$= 595$ sq. ft.

$\because$ the length of the room $= 34$ ft.,

$\therefore$ its reqd. breadth $= \frac{595}{34}$ ft. $= 17\frac{1}{2}$ ft.

Again, each boy requires $110\frac{1}{2}$ cu. ft. of space,

$\therefore$ The volume of the room $= 110\frac{1}{2}$ cu. ft. $\times 70 = 7735$ cu. ft.

but its length $= 34$ ft. and breadth $= \frac{35}{2}$ ft.

$\therefore$ the reqd. height $= \frac{7735 \times 2}{34 \times 35}$ ft. $= 13$ ft.

Ex. 10. The three conterminous edges of a rectangular solid are 36, 75 and 80 inches respectively ; find the edge of a cube which will be of the same capacity. [R. E.]

Here the dimensions of the rectangular solid are evidently 36 in., 75 in. and 80 in.

$\therefore$ The volume of the cube $=$ the volume of the solid

$= 36 \times 75 \times 80$ cu. in.

$\therefore$ The edge of the cube $= \sqrt[3]{36 \times 75 \times 80}$ in.

$= \sqrt[3]{3^3 \times 4^3 \times 5^3}$ in.

$= 3 \times 4 \times 5$ in. $= 60$ inches.

Ex. 11. A rectangular reservoir is 100 ft. long by 64 ft. broad ; at what rate of speed per hour must water flow into it through a pipe, whose cross-section is a square of side 2 inches, in order to make the water rise 2 ft. in 8 hours ? [B. U.]

Here, the length of the reservoir $= 110$ ft., and the breadth $= 64$ ft.

The water that flows into it in 8 hours is 2 feet deep.

$\therefore$ The volume of the water $=100$ ft. $\times 64$ ft. $\times 2$ ft.

$=12800$ cu. ft.

$\therefore$ The volume of water that enters in 1 hour $=\frac{12800}{8}$ cu. ft.

$=1600$ cu. ft.

Now, the cross-section of the pipe $=2$ in. $\times 2$ in. $=4$ sq. in.

$=\frac{1}{36}$ sq. ft.

$\therefore$ The speed of the flow of water $=(1600\div\frac{1}{36})$ ft. per hour

$=\frac{1600\times 36}{3\times 1760}$ m. p. h.

$=10\frac{10}{11}$ miles per hour.

Exercise 1

1. Find the total surface, the volume and the diagonal of a rectangular solid whose dimensions are 3 ft., 2 ft. and 4 ft.

2. Find the whole surface, the volume and the diagonal of a cube whose edge is 5 ft.

3. Find the total surface and the volume of a cube whose diagonal measures $3\sqrt{3}$ ft.

✓**4.** What is the length of the edge of a cube whose total surface is 37·5 sq. ft. ?

5. The dimensions of a rectangular solid are 8 ft., 6 ft. and 5 ft., find its total surface.

6. The diagonal of a cube is 30 in. ; what is the solid content ? [S. A.]

7. The dimensions of a rectangular solid are as 5 : 3 : 2 and its total surface measures 558 sq. in., find the dimensions.

8. The whole surface of a cube is equal to the sum of the whole surfaces of three other cubes whose edges are 3 in., 4 in. and 1 ft. respectively. Find the edge of the first cube.

9. The total surface of a rectangular solid of square base is 506 sq. ft. and its height is 6 ft., find its length and breadth.

PYRAMID

7. A **pyramid** is a solid bounded by plane faces, of which the one face called the **base** is a rectilineal figure, and the other faces are triangles having a common vertex at some point outside the plane of the base.

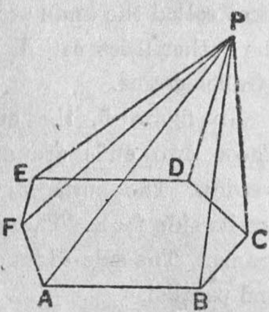


Fig. 7

Hence, the base of a pyramid is a triangle, or a quadrilateral or a polygon and its side faces are several triangles having the same vertex, which is also the vertex of the pyramid. [Fig. 7]

The perpendicular from the vertex to the base is the **height** of the pyramid.

The straight line in which any two triangles meet is called the edge of the pyramid.

If the perpendicular drawn from its vertex to its base passes through the central point of the base (*i.e.*, the centre of the circumscribed or the inscribed circle), then it is said to be a **right pyramid**. If the base of a right pyramid be a rectangle or a square, the perpendicular from the vertex meets the base at the point of intersection of its diagonals.

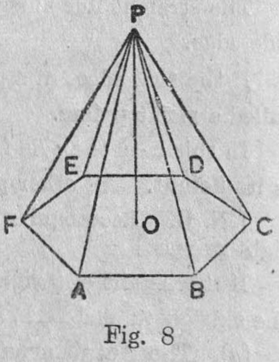


Fig. 8

If the base of a right pyramid be a regular figure, it is called a **regular** pyramid. Its side faces are congruent isosceles triangles.

Other pyramids are called oblique pyramids. [Fig. 9]. The perpendicular from the vertex to any side of the base of an oblique pyramid is called its **slant height.**

[N. B. In fig. 9, *PO* is the height and *PK* is the slant height of the pyramid.]

Tetrahedron : A tetrahedron is a pyramid on a triangular base. So it is bounded or contained by four triangular faces. [Fig. 10]

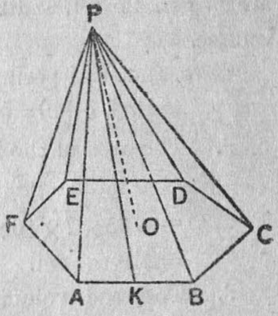


Fig. 9

When the base of a tetrahedron is an equilateral triangle, it is called a **right tetrahedron**.

A tetrahedron has four faces, four vertices and six edges.

When all the four faces are equilateral triangles, it is called a **regular** tetrahedron.

If the sides of the base of a right pyramid be a, b, c, d,...... units long, and the height be h units and the slant height be l units, then

(a) The total **area of the side faces** of a pyramid

$=\frac{1}{2}al+\frac{1}{2}bl+\frac{1}{2}cl+\frac{1}{2}dl+\ldots$

$=\frac{1}{2}(a+b+c+d+\ldots)l$

= **Semi-perimeter of the base × slant height.**

The **whole surface** =
the slant surfaces + the area of the base.

(b) The **volume** of a pyramid
$=\frac{1}{3}\times$ **area of base × height.**

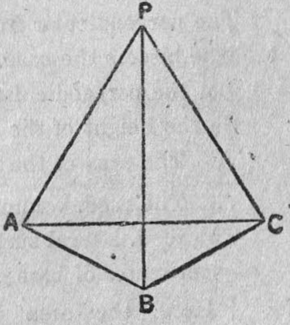


Fig. 10

Examples (2)

Ex. 1. The base of a right prism, 12 ft. high, is an equilateral triangle on a side of 3 ft. ; find the area of its rectangular faces.

The area of the rectangular faces = perimeter of the base × height.

Here, the perimeter of the base = the perimeter of the equilateral triangle = 3 ft. × 3 = 9 ft., and the height = 12 ft.

∴ The required area of the rectangular side faces
= 9 ft. × 12 ft. = 108 sq. ft.

✓**Ex. 2.** The base of a right prism is a triangle whose sides are 17 cm., 25 cm. and 28 cm. If its height is 20 cm., find its volume.

Here, the semi-perimeter (s) of the triangular base

$=\frac{1}{2}(a+b+c)=\frac{1}{2}(28+25+17)$ cm. $=35$ cm.

$\therefore$ The area of the base $=\sqrt{s(s-a)(s-b)(s-c)}$

$=\sqrt{35(35-28)(35-25)(35-17)}$ sq. cm.

$=\sqrt{35\times7\times10\times18}$ sq. cm. $=\sqrt{5^2\times6^2\times7^2}$ sq. cm.

$=210$ sq. cm.

$\therefore$ The reqd. volume of the right prism $=$ area of base $\times$ height

$=210$ sq. cm. $\times 20$ cm. $=4200$ cu. cm.

✓**Ex. 3.** Find the volume and lateral surface of a right prism 8 inches high standing on an isosceles triangle, each of whose equal sides is 5 inches and the other side 6 inches. [C. U. 1945]

The perpendicular from the vertex to the base of an isosceles triangle bisects the base.

Let the perpendicular be h. Then $h^2=5^2-3^2=16$, $\therefore$ $h=4$.

So the height of the triangle $=4$ inches.

$\therefore$ The area of the triangle $=\frac{1}{2}\times6$ in. $\times 4$ in. $=12$ sq. in.

$\therefore$ The reqd. volume $=12$ sq. in. $\times 8$ in. $=96$ cu. in.

The reqd. lateral surface of the right prism

$=$ perimeter of base $\times$ height $=(5+5+6)$ in. $\times 8$ in. $=128$ sq. in.

[Here, the area of the isosceles triangle can also be determined from the formula : area $=\sqrt{s(s-a)(s-b)(s-c.)}$]

✓**Ex. 4.** The base of a right prism, 10 inches high, is an equilateral triangle on a side of 8 in. ; find the volume of the prism.

The area of an equilateral triangle $=\frac{\sqrt{3}}{4}a^2$ (a being its side)

$\therefore$ The area of the base of the prism $=\frac{\sqrt{3}}{4}\times8^2$ sq. in.

$=16\sqrt{3}$ sq. in.

$\therefore$ The reqd. volume $=$ area of base $\times$ height

$=16\sqrt{3}$ sq. in. $\times 10$ in. $=160\sqrt{3}$ cu. inches.

Ex. 5. The volume of a right prism is 330 cu. cm. and its base is a triangle whose sides are 5 cm., 12 cm. and 13 cm. respectively. Find the height and the total surface of the prism.

Here, since $5^2+12^2=169=13^2$,

$\therefore$ the triangle is right-angled and the sides containing the right-angle are 5 cm. and 12 cm.

$\therefore$ The area of the base of the prism $=\frac{1}{2}\times 12\times 5$ sq. cm.
$=30$ sq. cm.

$$\therefore \text{ The reqd. height} = \frac{\text{volume}}{\text{area of base}} = \frac{330 \text{ cu. cm.}}{30 \text{ sq. cm.}} = 11 \text{ cm.}$$

Again, the perimeter of the base $=(5+12+13)$ cm. $=30$ cm.

$\therefore$ The area of the side faces $=30$ cm. $\times 11$ cm. $=330$ sq. cm.

$\therefore$ The total surface of the prism
$=$ area of the side faces $+$ area of the two ends
$=(330$ sq. cm. $+30$ sq. cm. $\times 2)=390$ sq. cm.

[**N. B.** You may apply the formula $\triangle=\sqrt{s(s-a)(s-b)(s-c)}$. The whole surface means the sum of the side faces and the two ends. The two ends $=$ base $\times 2$.]

Ex. 6. The area of the lateral surface of a right prism is 882 sq. cm. and its height 14 cm. ; if the base is a regular polygon of seven sides, find the length of each side.

Let each side of the base be a cm.

$\therefore$ The area of the side faces of the prism
$=$ perimeter of base $\times$ height $=7a\times 14$ cm. $=98a$ sq. cm.

$\therefore$ $98a=882$ (hyp.) $\therefore$ $a=\frac{882}{98}=9$.

$\therefore$ each side of the base $=9$ cm.

Ex. 7. The base of a right prism, $5\sqrt{3}$ ft, high, is a regular hexagon whose side is 6 ft. Find its volume.

Here, the base being a regular hexagon (*i. e.*, a polygon of 6 sides), the area of the base is equal to 6 times the area of an equilateral triangle whose each side is 6 ft.

$\therefore$ the area of the base of the prism $=\frac{\sqrt{3}}{4}\times 6^2\times 6$ sq. ft.

$=54\sqrt{3}$ sq. ft.

$\therefore$ its volume = area of the base × height

$=54\sqrt{3}$ sq. ft. $\times 5\sqrt{3}$ ft. $=810$ cu. ft.

✓**Ex. 8.** The base of a right prism is a trapezium whose parallel sides are 7 cm. and 13 cm., the perpendicular distance between those sides being 8 cm. If the height of the prism is 10 cm., find its volume.

The area of the trapezium $=\frac{1}{2}\times$ sum of the parallel sides × prep. distance $=\frac{1}{2}(7+13)\times 8$ sq. cm. $=80$ sq. cm.

$\therefore$ the volume of the prism = area of base × height

$=80$ sq. cm. $\times 10$ cm. $=800$ cu. cm.

[Pyramid]

✓**Ex. 9.** The base of a right pyramid, 6 in. high, is an equilateral triangle of side 4 inches. Find its volume and the area of the side faces.

Let P be the vertex and the equilateral $\triangle$ABC be the base of the pyramid. Draw AD$\perp$BC, then AD is the median of the $\triangle$ABC. Let O be the centroid of the triangle. $\therefore$ PO is perpendicular to the plane ABC. $\therefore$ PO is perpendicular to the median AD.

Suppose PD $=l$ and PO $=h$, then $h=6$ inches.

Now, $AD^2=AB^2-BD^2=4^2-2^2=12$, $\therefore$ AD $=\sqrt{12}$ in. $=2\sqrt{3}$ inches.

$\therefore$ OD $=\frac{1}{3}$AD $=\frac{2\sqrt{3}}{3}$ in. $=\frac{2}{\sqrt{3}}$ in.

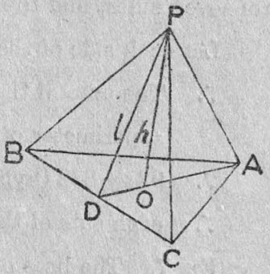


Fig. 11

$\therefore$ $l^2=h^2+OD^2=(36+\frac{4}{3})$ sq. in. $=\frac{112}{3}$ sq. in.

$\therefore$ $l=\sqrt{\frac{112}{3}}$ in. $=\frac{4\sqrt{7}}{\sqrt{3}}$ in.

$\therefore$ the volume of the pyramid $= \frac{1}{3} \times$ area of base $\times$ height

$$= \tfrac{1}{3} \times \tfrac{1}{2} \times 4 \times 2\sqrt{3} \times 6 \text{ cu. in.} = 8\sqrt{3} \text{ cu. inches.}$$

The area of the side faces $= \frac{1}{2} \times$ perimeter $\times$ slant height

$$= \frac{1}{2} \times 12 \text{ in.} \times \frac{4\sqrt{7}}{\sqrt{3}} \text{ in.} = 8\sqrt{21} \text{ sq. inches.}$$

Ex. 10. A right pyramid stands on a base 16 cm. square and its height is 15 cm. ; find its slant surfaces and the volume. [*H. S. '64(c)*]

Here the base is a square ABCD of side 16 cm. Let its diagonals AC and BD intersect at O and let OP be perpendicular to AB.

Then OP bisects AB, and OP = AP = 8 cm.

The height of the pyramid = 15 cm.

$\therefore$ its slant height $l = \sqrt{8^2 + 15^2}$ cm. = 17 cm.

$\therefore$ the area of its slant faces $= \frac{1}{2} \times$ perimeter $\times$ slant height

$$= \tfrac{1}{2} \times (16 \text{ cm.} \times 4) \times 17 \text{ cm.} = 544 \text{ sq. cm.}$$

Its volume $= \frac{1}{3} \times$ area of base $\times$ height

$$= \tfrac{1}{3}(16 \times 16) \text{ sq. cm.} \times 15 \text{ cm.} = 1280 \text{ cu. cm.}$$

Ex. 11. Determine the volume of a pyramid whose height is $10\sqrt{7}$ ft. and which stands on a triangle of sides 16 ft., 11 ft. and 9 ft. [C. U. '41]

The semi-perimeter of the triangle $(s) = \frac{1}{2}(16+11+9)$ ft. = 18 ft.

$\therefore$ the area of the triangle $= \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{18(18-16)(18-11)(18-9)} \text{ sq. ft.}$$

$$= \sqrt{18 \times 2 \times 7 \times 9} \text{ sq. ft.} = 18\sqrt{7} \text{ sq. ft.}$$

$\therefore$ the reqd. volume $= \frac{1}{3} \times$ area of base $\times$ height

$$= \tfrac{1}{3} \times 18\sqrt{7} \text{ sq. ft.} \times 10\sqrt{7} \text{ ft.} = 60 \times 7 \text{ cu. ft.} = 420 \text{ cu. ft.}$$

Ex. 12. A right pyramid stands on a square base of side 12 ft. Find the height of the pyramid, if its volume is 576 cu. feet. [C. U. '43]

Here, the area of the base of the pyramid

$$= 12 \text{ ft.} \times 12 \text{ ft.} = 144 \text{ sq. ft.},$$

$\therefore$ the required height

$$= \frac{\text{volume}}{\frac{1}{3} \times \text{area of base}} = \frac{576 \text{ cu. ft.}}{\frac{1}{3} \times 144 \text{ sq. ft.}} = 12 \text{ ft.}$$

✓**Ex. 13.** A pyramid on a square base has four equilateral triangles for its four other faces, each edge being 20 ft.; find the volume. [S. A.]

Here, the area of the base $= 20^2$ sq. ft. $= 400$ sq. ft.

The height is to be determined. Suppose the sq. ABCD is the base and P the vertex of the pyramid.

Let the diagonals AC and BD intersect at O. Then PO is the height of the pyramid.

Draw $OE \perp AB$, then $OE = AE = \frac{1}{2}AB = \frac{1}{2} \times 20$ ft. $= 10$ ft.

Now, OEP is a right-angled triangle. Its side PE = the height of the given equilateral triangle $= \underbrace{\frac{\sqrt{3}}{2} \times 20 \text{ ft.}} = 10\sqrt{3}$ ft.

$\therefore \quad PO^2 = PE^2 - OE^2 = (10\sqrt{3})^2 - 10^2 = 200,$

$\therefore \quad PO = \sqrt{200}$ ft. $= 10\sqrt{2}$ ft.

$\therefore$ the volume of the pyramid $= \frac{1}{3} \times$ area of base $\times$ height

$$= \frac{1}{3} \times 400 \text{ sq. ft.} \times 10\sqrt{2} \text{ ft.}$$

$$= \frac{4000\sqrt{2}}{3} \text{ cu. ft.}$$

$$= 1885{\cdot}6 \text{ cu. ft. (appx.)}$$

✓**Ex. 14.** The faces of a tetrahedron are four equal equilateral triangles. Find the area of the faces of the tetrahedron, if the length of a side of each triangle is 4 ft. Find also the volume of the tetrahedron. [C. U. '38]

Here, the slant height of the tetrahedron

= the height of the equilateral triangle of side 4 ft.

$$= \frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{2} \times 4 \text{ ft.} = 2\sqrt{3} \text{ ft.}$$

$\therefore$ the area of the side faces

$= \frac{1}{2} \times$ perimeter of base $\times$ slant height

$= \frac{1}{2} \times 12$ ft. $\times 2\sqrt{3}$ ft. $= 12\sqrt{3}$ sq. ft.,

and the area of the base $= \frac{\sqrt{3}}{4}a^2$ [a being the side of the $\triangle$]

$$= \frac{\sqrt{3}}{4} \times 4^2 \text{ sq. ft.} = 4\sqrt{3} \text{ sq. ft.}$$

$\therefore$ the reqd. area of the faces of the tetrahedron

$$=(12\sqrt{3}+4\sqrt{3})\text{ sq. ft.}=16\sqrt{3}\text{ sq. ft.}$$

Again, let the base be the equilateral $\triangle ABC$ and the vertex of the pyramid be P. Draw $AD \perp BC$, so that AD is a median of $\triangle ABC$. Let O be the centroid of $\triangle ABC$. $\therefore$ PO is perpendicular to the plane ABC, $\therefore$ PO is perp. to the median AD.

Let $PD=l=2\sqrt{3}$ ft. and $PO=h$.

Now, $DO=\frac{1}{3}AD=\frac{1}{3}\times\text{slant height}=\frac{1}{3}\times 2\sqrt{3}\text{ ft.}=\frac{2}{\sqrt{3}}\text{ ft.}$

$$\therefore \quad h^2=l^2-OD^2=(2\sqrt{3})^2-\left(\frac{2}{\sqrt{3}}\right)^2=12-\frac{4}{3}=\frac{32}{3},$$

$$\therefore \quad h=\sqrt{\frac{32}{3}}\text{ ft.}=\frac{4\sqrt{2}}{\sqrt{3}}\text{ ft.}=\frac{4}{3}\sqrt{6}\text{ ft.}$$

$\therefore$ the reqd. volume $=\frac{1}{3}\times$ area of base $\times$ height

$$=\frac{1}{3}\times 4\sqrt{3}\text{ sq. ft.}\times\frac{4}{3}\sqrt{6}\text{ ft.}=\frac{16}{3}\sqrt{2}\text{ cu. ft.}$$

Ex. 15. Find the cost of painting the vertical sides of a wooden column, 10 ft. high and standing on a regular nonagon whose side is 6 inches, at 2s. per sq. ft.

The base of the column is a regular polygon of 9 sides (nonagon), each side of which is 6 in. or $\frac{1}{2}$ ft.

$\therefore$ the perimeter of the base $=\frac{1}{2}\text{ ft.}\times 9=\frac{9}{2}\text{ ft.}$

$\therefore$ the area of the vertical faces $=\frac{9}{2}\text{ ft.}\times 10\text{ ft.}=45\text{ sq. ft.}$

$\therefore$ the required cost $=2\text{s.}\times 45=90\text{ s.}=$ £4. 10 s.

Exercise 2

1. The base of a right prism, 9 in. high, is an equilateral triangle whose side is 1 ft. 4 in. Find the area of its lateral surfaces.

2. The base of a right prism is a triangle whose each side measures 6 cm. If the height of the prism is $16\sqrt{3}$ cm., find its volume.

3. The base of a right prism is an equilateral triangle whose each side is a. If the height of the prism is h, shew that its volume is $\frac{\sqrt{3}}{4}a^2h$.

4. The volume of a right prism is 80 cu. ft. and its base is a triangle whose sides are 3 ft., 4 ft., and 5 ft. Find the height and the total surface of the prism. [H. S. '64]

5. The area of the lateral surface of a right prism is 1080 sq. ft. and its height is 12 ft. ; if the base is a regular nonagon, find the length of each side of the base.

6. Find the volume of a pyramid when its base is a regular hexagon, each side measuring 6 ft. and height 30 ft. [B. U.]

7. The base of a right prism is a trapezium whose parallel sides are 10 ft. and 12 ft. and the perpendicular distance between them is 10 ft. If the height of the prism is 8 ft., find its volume.

8. Find the total surface of a right prism, 18 ft. high and standing on a regular hexagon whose each side is 3 ft. long.

9. A pyramid on a square base has four equilateral triangles for its four other faces, each edge being 30 feet ; find the volume. [R. E.]

10. Find the area of the surface of a right prism, whose ends are squares of sides 3 inches, the height of the prism being one foot. [C. U. '36]

11. The base of a pyramid, 12ft. high, is a rectangle $10'$ by $15'$. Find the volume.

12. The base of a pyramid, 10 ft. high, is a regular hexagon of side 6 ft. Find the volume.

13. In a regular tetrahedron of side a, show that the height is equal to $\frac{1}{3}\sqrt{6}a$ and volume is equal to $\frac{1}{6\sqrt{2}}a^3$.

14. The base of a pyramid, 12 cm. high, is a triangle whose sides are 8 cm., 15 cm. and 17 cm. Find the volume of the pyramid. [C. U. '46, '48]

15. A right pyramid stands on a base 8 ft. square and its height is 3 ft. ; find its slant surface and volume.

16. The base of a right prism, 10 cm. high, is a triangle whose sides are 17 cm., 10 cm. and 9 cm. Find the volume and the whole surface of the prism. [C. U. '40]

17. The base of a right column, 12 ft. high, is a regular pentagon whose side is 4 inches. Find the cost of white-washing the vertical sides of the column at 4 annas per sq. ft.

18. A right pyramid stands on a rectangular base whose sides are 12 cm. and 9 cm. and the length of the slant edges is 8·5 cm. ; find the height and volume of the pyramid.
[H. S. '62 ; G. U. '48]

19. A right pyramid stands on a rectangular base whose sides are 24 cm. and 18 cm. and each of the slant edges is 17 cm. ; find the height and volume of the pyramid. [N. U. '47]

CONE

8. **A right circular cone :** The solid generated by the revolution of a right-angled triangle about one of the sides containing the right angle as axis is called a *right circular cone.*

In fig. 12, $\triangle ABC$ is right-angled at B. If the triangle revolves about the side AB as axis, the point C traces out a circle. This circle is the **base** of the cone. BC is the radius of the circle. The point A is the **vertex** of the cone and $\angle CAD$ is its vertical angle which is twice the angle A of $\triangle ABC$. AB is the **height** and AC or AD is the **slant height** of the cone.

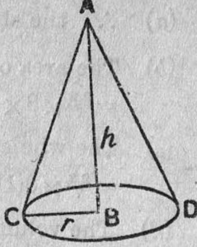


Fig. 12

The sharpened end of a pencil is an example of a cone.

Every plane section of a right circular cone parallel to the base is a circle ; and every section of the cone made by a plane through the axis is an isosceles triangle whose vertical angle is the vertical angle of the cone.

If h be the height, r the radius of the base and l the slant height of a right circular cone, then we have

(i) The **area of the slant surface of a right cone**

$=\frac{1}{2}$(circumference of the base) × slant height

$=\frac{1}{2}\times 2\pi r\times l=\pi \mathbf{rl}$ sq. units...(1).

or, $=\pi \mathbf{r}\sqrt{\mathbf{h}^2+\mathbf{r}^2}$ sq. units......(2).

[$\because$ $\angle ABC=1$ rt. angle, $\therefore AC^2=AB^2+BC^2, \therefore l^2=h^2+r^2$]

(ii) The **whole surface** of a cone

= area of its curved (slant) surface + area of its base

$=\pi rl+\pi r^2=\pi \mathbf{r}(l+\mathbf{r})$ sq. units

(iii) The **volume** of a right circular cone

$=\frac{1}{3}\times$ area of the base × height $=\frac{1}{3}\pi \mathbf{r}^2\mathbf{h}$ **cubic units.**

Examples (3)

[Assume $\pi=\frac{22}{7}$, if not otherwise stated]

Ex. 1. Find (a) the slant height, (b) the curved surface, (c) the whole surface and (d) the volume of the cone whose height is 15 inches and the diameter of whose base is 16 inches.

Here, the height $h=15$ in., and radius $r=8$ in.

(a) $\therefore$ the slant height $(l)=\sqrt{h^2+r^2}=\sqrt{15^2+8^2}$ in. $=17$ in.

(b) The area of the curved surface $=\pi rl$ sq. units

$=\frac{22}{7}\times 8\times 17$ sq. in. $=427\frac{3}{7}$ sq. inches.

(c) The whole surface of the cone $=\pi r(l+r)$ sq. units

$=\frac{22}{7}\times 8(17+8)$ sq. in. $=628\frac{4}{7}$ sq. inches.

(d) The volume of the cone $=\frac{1}{3}\pi r^2 h$ cu. units

$=\frac{1}{3}\times\frac{22}{7}\times 8^2\times 15$ cu. in.

$=1005\frac{5}{7}$ cu. inches.

Ex. 2. Find the vertical height and the slant height of a right circular cone whose curved surface is 704 sq. inches and the diameter of whose base is 2 ft. 4 inches.

Here the radius (r) of the base $= 2$ ft. 4 in. $\times \frac{1}{2} = 14$ in.

$\because$ the area of the curved surface of a cone $= \pi r l$,

$\therefore$ the reqd. slant height $= \dfrac{\text{curved surface}}{\pi r} = \dfrac{704}{\frac{22}{7} \times 14}$ in. $= 16$ in.

Again, $\because l^2 = h^2 + r^2, \quad \therefore h^2 = l^2 - r^2, \quad \therefore h = \sqrt{l^2 - r^2}$.

$\therefore$ The reqd. vertical height $= \sqrt{h^2 - r^2} = \sqrt{16^2 - 14^2}$ in.

$= \sqrt{60}$ in. $= 2\sqrt{15}$ in.

Ex. 3. The slant height of a right circular cone is 1 ft. 9 in. and its curved surface is 396 sq. in.; find the diameter of the base.

Here the slant height (l) $= 1$ ft. 9 in. $= 21$ in.

$\because$ The area of the slant surface $= \pi r l$,

$\therefore \quad r = \dfrac{\text{area of slant surface}}{\pi l} = \dfrac{396 \text{ sq. in.}}{\frac{22}{7} \times 21 \text{ in.}} = 6$ in.

$\therefore$ The reqd. diameter $= 2r = 12$ in. $= 1$ foot.

Ex. 4. The volume of a cone is 154 cu. ft. and its height is 12 ft. Find the radius of the base.

$\because \frac{1}{3}\pi r^2 h =$ the volume of a cone,

$\therefore$ here $\frac{1}{3} \times \frac{22}{7} \times r^2 \times 12$ ft. $= 154$ cu. ft.

$\therefore \quad r^2 = \frac{154 \times 3 \times 7}{22 \times 12}$ sq. ft. $= \frac{49}{4}$ sq. ft., $\quad \therefore \quad r = \frac{7}{2}$ ft.

$\therefore$ The reqd. radius $= 3$ ft. 6 in.

Ex. 5. The height of a conical tent is $7\frac{1}{2}$ ft. and it is to enclose 200 sq. yds. of ground. Find how much canvas will be required. [R. M. A.]

Here, the area of the circular base $= 200$ sq. yds.,

i.e., $\pi r^2 = 200$ sq. yds., or, $\frac{22}{7} r^2 = 1800$ sq. ft., or, $r^2 = \frac{6300}{11}$ sq. ft., $\therefore$ $r = 30\sqrt{\frac{7}{11}}$ ft.

$\because$ $l^2 = r^2 + h^2$, $\therefore$ here $l^2 = \frac{6300}{11} + (\frac{15}{2})^2 = \frac{27675}{44}$ sq. ft.

$\therefore$ $l = \sqrt{\frac{27675}{44}}$ ft. $= \frac{15}{2}\sqrt{\frac{123}{11}}$ ft.

$\therefore$ Canvas required = area of the curved surface of the tent

$$= \pi r l = \frac{22}{7} \times \frac{30\sqrt{7}}{\sqrt{11}} \times \frac{15 \times \sqrt{123}}{2 \times \sqrt{11}} \text{ sq. ft.}$$

$$= \frac{30 \times 15\sqrt{7} \times \sqrt{123}}{7} \times \frac{1}{9} \text{ sq. yd.}$$

$$= \tfrac{50}{7}\sqrt{861} \text{ sq. yd.} = 209{\cdot}5 \text{ sq. yds. (App.)}.$$

Ex. 6. A right-angled triangle, of which the sides are 3 ft. 6 in. and 5 ft. in length, is made to turn round on the longer side ; find the volume of the solid thus formed. [S. A.]

Here, since the triangle is revolving about the side, 5 ft. long as axis, the height of the cone formed is 5 ft.

The other given side is evidently the radius of the base of the cone and is equal to $\frac{7}{2}$ ft.

$\therefore$ The reqd. volume $= \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times (\frac{7}{2})^2 \times 5$ cu. ft.

$= \frac{385}{6}$ cu. ft. $= 64\frac{1}{6}$ cu. ft.

Ex. 7. The sides of a right-angled triangle are 3 in. and 4 in. respectively ; find the volume of the cone formed by the revolution of the triangle round the hypotenuse. [S. A.]

[Draw a fig.]

Let the angle B of $\triangle$ABC be a right angle, and let AB = 4 in. and BC = 3 inches.

Then $AC = \sqrt{AB^2 + BC^2} = \sqrt{4^2 + 3^2}$ in. = 5 in.

Draw $BD \perp AC$. $\because$ BD is drawn perpendicular from the right angle to the hypotenuse of a right-angled triangle,

$$\therefore \quad \frac{BD}{AB} = \frac{BC}{AC}, \text{ or, } \frac{BD}{4} = \frac{3}{5}, \quad \therefore \; BD = \frac{12}{5} \text{ inches.}$$

Now, two cones are generated when the $\triangle$ABC revolves round AC. BD is the radius of the base and AD the height of one cone, and BD is the radius and CD the height of the other.

$\therefore$ The total volume of the two cones

$$= \tfrac{1}{3}\pi \times BD^2 \times AD + \tfrac{1}{3}\pi \times BD^2 \times CD$$
$$= \tfrac{1}{3}\pi \times BD^2(AD + CD) = \tfrac{1}{3}.\pi.BD^2.AC$$
$$= \tfrac{1}{3} \times \tfrac{22}{7} \times (\tfrac{12}{5})^2 \times 5 \text{ cu. in.} = \tfrac{1056}{35} \text{ cu. in.}$$
$$= 30\tfrac{6}{35} \text{ cu. inches.}$$

Ex. 8. A right-angled triangle, whose remaining angles are 60° and 30°, revolves about its hypotenuse, which is 12 in. long; find the volume of the solid thus formed.

$\because$ One acute angle of the right-angled triangle is double the other acute angle,

$\therefore$ the least side (*i. e.*, the side opposite to angle 30°) is half of the hypotenuse.

[Draw a fig. as in. Ex. 7] Here BC = 6 in., AC = 12 in.

$\therefore$ $AB^2 = AC^2 - BC^2 = (12^2 - 6^2)$ sq. in. $= 108$ sq. in.

$\therefore$ $AB = \sqrt{108}$ in. $= 6\sqrt{3}$ in.

$\because$ $\dfrac{BD}{AB} = \dfrac{BC}{AC}$, $\therefore$ here $\dfrac{BD}{6\sqrt{3}} = \dfrac{6}{12}$, $\therefore$ $BD = 3\sqrt{3}$.

$\therefore$ The reqd. volume $= \tfrac{1}{3}\pi \times BD^2(AD + DC)$

$$= \tfrac{1}{3}\pi \times BD^2 \times AC = \tfrac{1}{3} \times \tfrac{22}{7} \times (3\sqrt{3})^2 \times 12 \text{ cu. in.}$$
$$= \tfrac{2376}{7} \text{ cu. in.} = 339\tfrac{3}{7} \text{ cu. inches.}$$

Ex. 9. A conical tent is required to accommodate 5 people; each person must have 16 sq. ft. of space on the ground, and 100 cu. ft. of air to breathe; give the vertical height, slant height and the width of the tent. [R. U. S.]

Here, the width of the tent or diameter of the base (or $2r$), its height (h) and slant height (l) are to be determined.

Now, the space required for 5 persons $= 16$ sq.ft. $\times 5 = 80$ sq. ft.

$\therefore$ Area of the base, *i.e.*, $\pi r^2 = 80$ sq. ft., or, $\tfrac{22}{7}r^2 = 80$ sq. ft.

or, $r^2 = \tfrac{80 \times 7}{22}$ sq. ft. $= \tfrac{280}{11}$ sq. ft., $\therefore$ $r = \sqrt{\tfrac{280}{11}}$ ft. $= 5{\cdot}045$ ft.

$\therefore$ The reqd. width of the tent $= 2r = 5{\cdot}045$ ft. $\times 2$

$$= 10{\cdot}09 \text{ ft. (App.).}$$

Again, the volume of the tent $=100$ cu. ft. $\times 5=500$ cu. ft.

i.e., $\frac{1}{3}\pi r^2 h=500$ cu. ft., or, $\frac{1}{3}\times\frac{22}{7}\times\frac{280}{11}\times h=500$ cu. ft.

$\therefore \quad h=\frac{500\times 3\times 7\times 11}{22\times 280}$ ft. $=\frac{75}{4}$ ft. $=18\frac{3}{4}$ ft.

Now, $\because \; l^2=r^2+h^2=\frac{280}{11}+(\frac{75}{4})^2=\frac{66355}{176}$ sq. ft.

$\therefore \quad l=\sqrt{\frac{66355}{176}}$ ft. $=19{\cdot}4$ ft. (App.).

Ex. 10. Divide a right circular cone into two parts by a plane drawn parallel to the base so that the curved surfaces of the two parts are equal.

Let APB be a right circular cone, the circle AB its base, O the centre and r the radius of the base.

Suppose a plane, drawn through the pt. C in AP parallel to the base, divides the cone into two parts so that the curved surfaces of the two parts are equal. Then the curved surface of the cone CPD is equal to that of the remaining part ACDB, *i.e.*, the area of the curved surface of the cone CPD is half the area of the curved surface of the cone APB.

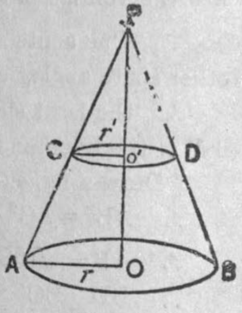


Fig. 13

The parallel plane drawn throught C parallel to the base cuts the cone in a circle. Let O′ and r' be respectively the centre and radius of this circle.

Now, the $\triangle$s PCO′, PAO are similar ($\because$ CO′ $\parallel$ AO).

$$\therefore \quad \frac{CO'}{AO}=\frac{PC}{PA}, \; i.e., \; \frac{r'}{r}=\frac{PC}{PA},$$

$$\therefore \quad \frac{\text{the area of the curved surface of the cone CPD}}{\text{area of the curved surface of the cone APB}}$$

$$=\frac{\pi r'.\,PC}{\pi r.\,PA}=\frac{r'}{r}.\frac{PC}{PA}=\frac{PC}{PA}.\frac{PC}{PA}=\frac{PC^2}{PA^2},$$

$$\therefore \quad \frac{PC^2}{PA^2}=\frac{1}{2}, \quad \therefore \quad PC^2=\tfrac{1}{2}PA^2=PA.\tfrac{1}{2}PA.$$

$\therefore$ The plane is to be drawn at C in AP so thet PC is the mean proportional between PA and $\frac{1}{2}$ PA.

✓ **Ex. 11.** A right circular cone, 20 ft. high, has its upper part cut off by a plane passing through the middle point of its axis. If the plane of section be at right angles to the axis and if the radius of the original cone be 4 ft., find the volume of the truncated cone. [C. U. 1936]

The volume of the given cone $= \frac{1}{3}\pi r^2 h$

$$= \tfrac{1}{3}\pi \times 4^2 \times 20 \text{ cu. ft.} = \tfrac{320}{3}\pi \text{ cu. ft.}$$

Since the cone is cut by a plane through the middle point of its axis, its upper part is a cone whose height is $\frac{1}{2} \times 20$ ft. or 10 ft. and radius of whose base is $\frac{1}{2} \times 4$ ft. or 2 ft.

$\therefore$ the volume of the upper part $= \frac{1}{3}\pi \times 2^2 \times 10$ cu. ft.

$$= \tfrac{40}{3}\pi \text{ cu. ft.}$$

Hence, the required area of the truncated cone

$$= (\tfrac{320}{3}\pi - \tfrac{40}{3}\pi) \text{ cu. ft.} = \tfrac{280}{3} \times \tfrac{22}{7} \text{ cu. ft.} = 293\tfrac{1}{3} \text{ cu. ft.}$$

Exercise 3

1. The height and the diameter of the base of a right circular cone are 12 in. and 10 in. respectively. Find (1) the slant height, (2) the curved surface, (3) the whole surface and (4) the volume of the cone.

2. Find the vertical height and the slant height of a right circular cone whose curved surface is 330 sq. in. and the diameter of whose base is 7 inches.

3. The height and radius of the base of a cone are 12 cm. and 5 cm. respectively. Find the curved surface and the volume of the cone.

4. Find the radius of the base of a right circular cone whose volume is 1232 cu. ft. and height 24 ft.

5. The slant height of a right circular cone is 1 ft. 2 in. and its curved surface is 264 sq. in. Find the diameter of the base.

6. The height and the radius of the base of a cone are 15 cm. and 8 cm. respectively ; find its curved surface and the volume of the cone.

7. How much canvas will make a conical tent 11 ft. in height and 12 ft. in diameter at the base ?

8. A right-angled triangle, of which the sides are 3 in. and 4 in. in length, is made to turn round on the longer side. Find the volume of the cone thus formed. [S. A.]

9. A right-angled triangle of sides equal to 20 in., 16 in. and 12 in. respectively is made to spin round on its hypotenuse as axis. Find the volume of the double cone thus formed. [R. E.]

10. A cone 3 ft. high and 2 ft. in diameter at the bottom is placed on the ground and sand is poured over it until a conical heap is formed 5 ft. high and 30 ft. in circumference at the bottom. Find how many cubic feet of sand there are.

11. Find the lateral surface and the volume of a right circular cone of height 15 ft., the radius of whose base is 8 ft.

[C. U. '42]

12. Find the volume and the area of the slanting surface of a right circular cone of height 4 ft. and the radius of whose base is 3 feet. ($\pi = \frac{22}{7}$). [C. U. '39]

[**N. B.** For Circular Cylinder and Sphere see Appendix.]

CHAPTER II

ALGEBRA

ELIMINATION

1. Elimination and **Eliminant**. The **elimination** of one or more algebraical quantities from some given equations is the process of obtaining from those equations a new equation, free from those quantities and containing only the other quantities of the equations.

The equation thus obtained is called the **eliminant** of the given equations.

2. Process of Elimination: Suppose x is to be eliminated from the equations $2x+a=0$ and $3x+4b=0$.

Here, $\because\ 2x+a=0,\ \therefore\ x=-\frac{a}{2}$. Substituting this value of x in the other equation, we have $3\times-\frac{a}{2}+4b=0$,

or, $-3a+8b=0$, or, $3a-8b=0$.

Now, this equation $3a-8b=0$ is free from x (*i. e.*, there is no x in it) and it is derived from the given equations and involves only the other quantities a and b of the given equations.

Hence, $3a-8b=0$ is the required eliminant. It is called the **x-eliminant** of the given equations.

To eliminate one quantity (x or y) from two equations, first find the value of that quantity from one equation, and then substitute the value obtained in the other. The equation thus obtained is the eliminant of that quantity of the equations.

The elimination is often effected by applying various artifices and not by the above method.

The **number of equations in elimination**: It appears from the above example that two equations are necessary for eliminating one quantity (x). Generally the number of equations should be one more than the number of quantities to be eliminated. Thus to eliminate n quantities, $n+1$ equations are necessary. For, we require n equations to obtain the values of the n quantities and then get the eliminant by substituting those values in the remaining equation.

But if the given equations be **homogeneous equations** in the quantities to be eliminated, the number of equations required may be equal to the number of those quantities.

For example, suppose x and y are to be eliminated from $4x+my=0$ and $nx+3y=0$.

Now, dividing the equations by y we have

$$4.\frac{x}{y}+m=0\ldots(1) \text{ and } n.\frac{x}{y}+3=0\ldots(2).$$

From (1) and (2) we have $\frac{x}{y}=-\frac{m}{4}$ and $\frac{x}{y}=-\frac{3}{n}$,

$\therefore\ -\frac{m}{4}=-\frac{3}{n}$ or, $mn=12$. $\therefore$ the eliminant is $mn=12$.

[Here $\frac{x}{y}$ is taken as one quantity and eliminated.]

Examples (1)

Ex. 1. Eliminate x from the equations $a_1x+b_1=0$ and $a_2x+b_2=0$.

$$\because\ a_1x+b_1=0,\quad \therefore\ x=-\frac{b_1}{a_1}.$$

Again, $\because\ a_2x+b_2=0,\ \therefore\ x=-\frac{b_2}{a_2}$.

$\therefore\ -\frac{b_1}{a_1}=-\frac{b_2}{a_2},\ \therefore\ a_1b_2-b_1a_2=0$ is the reqd. eliminant.

Ex. 2. Eliminate x and y from the equations $a_1x+b_1y=0$ and $a_2x+b_2y=0$.

Dividing the given equations by y, we have $a_1\dfrac{x}{y}+b_1=0$...(1) and $a_2\dfrac{x}{y}+b_2=0$...(2). From (1) we have $\dfrac{x}{y}=-\dfrac{b_1}{a_1}$.

$\therefore$ Substituting this value of $\dfrac{x}{y}$ in (2) we have

$$a_2\left(-\frac{b_1}{a_1}\right)+b_2=0, \quad \text{or,} \quad a_1b_2-a_2b_1=0.$$

Ex. 3. Eliminate x, y, z from the equations

$$\left.\begin{array}{l} a_1x+b_1y+c_1z=0\ldots\ldots(1) \\ a_2x+b_2y+c_2z=0\ldots\ldots(2) \\ a_3x+b_3y+c_3z=0\ldots\ldots(3) \end{array}\right\}$$

By the method of cross-multiplication we have from (2) and (3)

$$\frac{x}{b_2c_3-b_3c_2}=\frac{y}{c_2a_3-c_3a_2}=\frac{z}{a_2b_3-a_3b_2}.$$

Let each ratio $=k$.

$\therefore \quad x=k(b_2c_3-b_3c_2),\ y=k(c_2a_3-c_3a_2),\ z=k(a_2b_3-a_3b_2).$

Now, putting the values of x, y, z in (1), we have

$$k\{a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2)+c_1(a_2b_3-a_3b_2)\}=0,$$

$$\therefore \quad a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2)+c_1(a_2b_3-a_3b_2)=0.$$

[**N. B.** Given equations may be reduced to the above given forms.]

Ex. 4. Eliminate m and n from the equations $mx+ny=a$, $nx-my=b$, $m^2+n^2=1$.

Squaring the first two given equations we have

$$m^2x^2+n^2y^2+2mnxy=a^2\ldots\ldots(1)$$

and $n^2x^2+m^2y^2-2mnxy=b^2\ldots\ldots(2)$

$\therefore$ adding (1), (2) we have $x^2(m^2+n^2)+y^2(n^2+m^2)=a^2+b^2$,

$\therefore$ the eliminant is $x^2+y^2=a^2+b^2 \quad [\because\ m^2+n^2=1\ ].$

Ex. 5. Eliminate p from the equations

$$x^2=p^2+\frac{1}{p^2} \text{ and } y=p+\frac{1}{p}.$$

Here $y^2=\left(p+\frac{1}{p}\right)^2=p^2+\frac{1}{p^2}+2=x^2+2, \therefore\ y^2-x^2=2.$

$\therefore$ the reqd. eliminant is $y^2-x^2=2$.

Ex. 6. Eiminate x from the equation $ax^2+bx+c=0$ and $a_1x^2+b_1x+c_1=0$.

By the method of cross-multiplication we have from the given equations

$$\frac{x^2}{bc_1-b_1c}=\frac{x}{ca_1-c_1a}=\frac{1}{ab_1-a_1b},$$

$$\therefore\quad x^2=\frac{bc_1-b_1c}{ab_1-a_1b} \text{ and } x=\frac{ca_1-c_1a}{ab_1-a_1b},$$

$$\therefore\quad \frac{bc_1-b_1c}{ab_1-a_1b}=\frac{(ca_1-c_1a)^2}{(ab_1-a_1b)^2},$$

$$\therefore\quad (ca_1-c_1a)^2=\frac{bc_1-b_1c}{ab_1-a_1b}\times(ab_1-a_1b)^2$$

$=(bc_1-b_1c)(ab_1-a_1b)$, which is the reqd. eliminant.

Ex. 7. Eliminate x and y from the equations

$x-y=a$......(1), $x^2-y^2=b^2$......(2) and $x^3-y^3=c^3$......(3).

$\because\ x^2-y^2=b^2, \quad \therefore\ (x+y)(x-y)=b^2,$

or, $(x+y)\times a=b^2, \quad \therefore\ x+y=\frac{b^2}{a}$......(4).

From (1) we have $x^2+y^2-2xy=a^2$......(5)

and from (4) we have $x^2+y^2+2xy=\frac{b^4}{a^2}$......(6)

From (6) $-$ (5) we have $4xy=\frac{b^4-a^4}{a^2}, \quad \therefore\ xy=\frac{b^4-a^4}{4a^2}.$

Now, from (3) we have $c^3=x^3-y^3=(x-y)(x^2+xy+y^2)$

$$=a\{(x+y)^2-xy\}$$

$$=a\left\{\frac{b^4}{a^2}-\frac{b^4-a^4}{4a^2}\right\}=a\left(\frac{3b^4+a^4}{4a^2}\right)=\frac{3b^4+a^4}{4a},$$

$\therefore\quad 3b^4+a^4=4ac^3.$

$\therefore\quad$ the reqd. eliminant is $a^4+3b^4-4ac^3=0.$

✓**Ex. 8.** Eliminate $\cos\theta$ and $\sin\theta$ between the equations

$$x\cos\theta+y\sin\theta=a \text{ and } x\sin\theta-y\cos\theta=b.$$

Here $\because\quad x\cos\theta+y\sin\theta=a,$

$$\therefore\quad x^2\cos^2\theta+y^2\sin^2\theta+2xy\cos\theta\sin\theta=a^2\text{......(1)}$$

Again, $\because\quad x\sin\theta-y\cos\theta=b,$

$$\therefore\quad x^2\sin^2\theta+y^2\cos^2\theta-2xy\sin\theta\cos\theta=b^2\text{......(2)}$$

Now, adding (1) and (2) we have

$$x^2(\sin^2\theta+\cos^2\theta)+y^2(\sin^2\theta+\cos^2\theta)=a^2+b^2$$

$$\therefore\quad x^2+y^2=a^2+b^2\quad(\because\quad \sin^2\theta+\cos^2\theta=1).$$

✓**Ex. 9.** Show that $p^2+q^2+r^2-4=pqr$ is the eliminant of the equations $\frac{y}{z}+\frac{z}{y}=p,\quad \frac{z}{x}+\frac{x}{z}=q,\quad \frac{x}{y}+\frac{y}{x}=r.$

Multiplying the given equations we have

$$2+\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)+\left(\frac{z^2}{y^2}+\frac{y^2}{z^2}\right)+\left(\frac{x^2}{z^2}+\frac{z^2}{x^2}\right)=pqr,$$

or, $2+\left(\frac{x}{y}+\frac{y}{x}\right)^2-2+\left(\frac{z}{y}+\frac{y}{z}\right)^2-2+\left(\frac{x}{z}+\frac{z}{x}\right)^2-2=pqr,$

or, $2+r^2-2+p^2-2+q^2-2=pqr,$

$\therefore\quad p^2+q^2+r^2-4=pqr.$

Hence it is proved that the eliminant of the given equations is $p^2+q^2+r^2-4=pqr.$

✓**Ex. 10.** Find the eliminant of the following equations

$$\left.\begin{aligned} x+y+z&=a\text{......(1)}\\ xy+yz+zx&=b\text{......(2)}\\ x^3+y^3+z^3&=c\text{......(3)}\\ xyz&=d\text{......(4)}\end{aligned}\right\}$$

From (3) and (4) we have $x^3+y^3+z^3-3xyz=c-3d$.

$\therefore$ $(x+y+z)(x^2+y^2+z^2-xy-yz-zx)=c-3d$,

or, $a\{(x+y+z)^2-3(xy+yz+zx)\}=c-3d$,

or, $a(a^2-3b)=c-3d$,

$\therefore$ $a^3-3ab+3d-c=0$, which is the reqd. eliminant.

Ex. 11. Eliminate x from $ax^2+bx+c=0$......(1) and $x^3+x+d=0$......(2).

Multiplying (1) by x and (2) by a we have

$$ax^3+bx^2+cx=0\text{......(3)}$$

$$ax^3+ax\ +ad=0\text{......(4)}$$

Subtracting (4) from (3) we have $bx^2+(c-a)x-ad=0$......(5).

Now, from(1) and(5) we have by the rule of cross-multiplication

$$\frac{x^2}{-abd-c(c-a)}=\frac{x}{bc+a^2d}=\frac{1}{a(c-a)-b^2},$$

$\therefore$ $(bc+a^2d)^2=(abd+c^2-ac)(a^2+b^2-ac)$. [See Ex. 6]

Exercise 1

Eliminate x from the following equations :—

1. $\left.\begin{array}{r}x+b=0\\3x+2a=a\end{array}\right\}$ **2.** $\left.\begin{array}{r}2x-m=0\\nx-5=0\end{array}\right\}$

3. $ax+b=0,\ a'x+b'=0$.

4. $ax^2-2a^2x+1=0,\ a^2+x^2=3ax$.

5. $x^2+ax+b=0,\ x^3+cx+d=0$.

6. $x+\dfrac{1}{x}=a+b,\qquad x-\dfrac{1}{x}=a-b$.

7. $x^2+x+a=0,\ bx+c=0$.

8. $x^3+\dfrac{1}{x^3}+3\left(x+\dfrac{1}{x}\right)=a,\quad x^3-\dfrac{1}{x^3}-3\left(x-\dfrac{1}{x}\right)=b$.

Eliminate x and y from the following equations :—

9. $2x+ay=0,\ bx+3y=0$.

10. $x-y=a,\ x^2+y^2=b,\ xy=c$.

11. $x^2-y^2=ax-by,\ x^2+y^2=1,\ 4xy=bx+ay.$

12. $x+y=p,\ x^2+y^2=q,\ x^3+y^3=r.$

13. $x+y=m,\ x^3+y^3=n,\ xy=l.$

14. $x+y=a,\ x^2+y^2=b^2,\ x^4+y^4=c^4.$

Eliminate x, y, z from the following equations :—

15. $ax+by+cz=0,\ a_1x+b_1y+c_1z=0,\ a_2x+b_2y+c_2z=0.$

16. $bx+ay-z=0,\ cx+az-y=0,\ cy+bz-x=0.$

17. $xy=c^2,\ yz=a^2,\ zx=b^2,\ x^2+y^2+z^2=d^2.$

18. $\dfrac{x}{a}=y+z,\ \dfrac{y}{b}=z+x,\ \dfrac{z}{c}=x+y.$

19. Eliminate m and n from the equations

$mx-ny=a(m^2-n^2),\ nx+my=2amn,\ m^2+n^2=1.$

20. Show that $a^2+b^2+c^2-abc=4$ is the eliminant of

$x^2+y^2=cxy,\ y^2+z^2=ayz,\ z^2+x^2=bzx.$

21. Eliminate l and m from the equations

$l^3x+m^3y=a,\ l^2+m^2=1$ and $lx-my=0.$ [P. U. 1902]

22. Eliminate a, b, c from the equations

$bz+cy=a,\ cx+az=b,\ ay+bx=c.$ [C. U., A. U.]

PROGRESSION

3. If a collection of numbers or quantities be such that each is formed from the preceding one by some fixed law, it is called a **series.** Thus the numbers 3, 5, 7, 9...... ; or 3, 6, 12, 24,...form a series. Each term in the first series is formed by adding 2 to its preceding term, while in the second series each term is formed by multiplying the preceding one by 2.

ARITHMETICAL PROGRESSION

4. A series is said to be in Arithmetical Progression (or, in A. P.), if the algebraic difference between any two successive terms be the same throughout the series.

This difference is called the **common difference** (c.d.)

Thus, (i) 1, 3, 5, 7,...are in Arithmetical Progression, as here the common difference throughout is 2. (ii) 10, 7, 4, 1...are in A. P., as here the common difference throughout is -3. (iii) $a, a+b, a+2b, \ldots$form an A. P.,as here the c.d. is b throughout.

[**N. B.** It is evident that each term of a series in A. P. is formed by adding a constant quantity (the common difference) to its preceding term.]

Common Difference (c. d.) : The common difference of an A. P. is found by subtracting any term from its succeeding term. Generally it is found by subtracting the first term from the second term.

Thus, (1) In the A. P. 2, 6, 10, 14,..., the common difference $=6-2=4$;

(2) In the series 7, 4, 1, -2...,the c. d.$=4-7=-3$;

(3) In $a, a+b, a+2b, \ldots$,the c.d.$=(a+b)-a=b$;

(4) In $a, a-b, a-2b, \ldots$, the c.d.$=(a-b)-a=-b$.

5. General Term, Last Term and the Number of Terms :

Let 4, 6, 8, 10, 12 and 14 be a series in A. P. Its last term is evidently 14, and number of its terms$=6$. Here 14 may also be called the sixth term of the series.

If there be several terms in an A.P., the nth term, of the series is called its general term, while the 4th, 5th, 10th, etc. terms are particular terms.

If, however, the series ends with the nth term, then the nth term is also the last term, and the number of terms of the series is n. This n (number of terms) must always be a positive integer, it cannot be a fraction or a negative integer.

Generally, the first term of an A. P. is denoted by a, the common difference by b (or, d), the last term by l, the number of terms by n and the sum of the series by s.

Again the first term is indicated by t_1, the second term by t_2, third term by $t_3, \ldots$, nth term by t_n.

5 (a). To find the nth term of an A. P.

Let a be the first term and b the common difference of an A.P.

Then t_1 (first term)$=a=a+(1-1)\,b$,

t_2 (2nd term)$=a+b=a+(2-1)\,b$,

t_3 (3rd term)$=a+2b=a+(3-1)\,b$,

t_4 (4th term)$=a+3b=a+(4-1)\,b$, and so on.

So it is evident that in any particular term the coefficient of b is one less than the serial number of the term.

$\therefore$ In any A. P. the **(nth term)** $\mathbf{t_n=a+(n-1)}\,b$.

If there be n terms in the series and l be the last term, then we have $l=\mathbf{a+(n-1)b}$.

Examples (2)

Ex. 1. Find the 12th term of an A. P. whose first term is 3 and the common difference is 2.

Here a (1st term)$=3$, b (common difference)$=2$ and n (number of terms)$=12$. $\because$ $t_n=a+(n-1)b$,

$\therefore$ the reqd. $t_{12}=3+(12-1)\times 2=3+22=25$.

Ex. 2. Find the 15th term of the series, 6, 8, 10,...

Here, the 1st term $a=6$, common difference $b=8-6=2$, n (number of terms)$=15$.

Now, from the formula $t_n=a+(n-1)\,b$ we have,

$$t_{15}=6+(15-1)\times 2=6+28=34.$$

Ex. 3. Find the 20th term of the series 10, 7, 4,...

Here $a=10$; b (com. diff.)$=7-10=-3$, $n=20$.

$\therefore$ $t_{20}=a+(n-1)b=10+(20-1)\times -3=10+19\times -3$.

$=10-57=-47$.

Ex. 4. The first term of an A. P. is 1 and the 18th term is 52 : find the common difference.

Let b be the common difference.

Here $t_{18}=52$, *i. e*,, $a+(18-1)b=52$,

or, $1+17b=52$ [$\because a=1$], or, $17b=51$, $\therefore$ $b=3$.

$\therefore$ the required common difference$=3$.

Ex. 5. Find the 21st term of a series whose 7th and 13th terms are 23 and 41 respectively.

Let a be the 1st term and b the common difference.

Here $t_7=23$, *i.e.*, $a+6b=23 \ldots (1)$

and $t_{13}=41$, *i. e.*, $a+12b=41 \ldots (2)$.

Now, subtracting (1) from (2) we get $6b=18$, $\therefore \quad b=3$.

Again, from (1) we have $a+18=23$, $\therefore \quad a=5$.

$\therefore \quad t_{21}=a+20b=5+20\times 3=5+60=65$.

Ex. 6. Find the series whose 5th and 12th terms are 8 and -6 respectiveiy.

Suppose $t_1=a$, and com. diff. $=b$.

$$\begin{aligned} \therefore \quad \text{From the given conditions } a+4b &= 8 \ldots (1) \\ \text{and } \quad a+11b &= -6 \ldots (2) \\ \hline \therefore \quad \text{(sub.)} \quad -7b &= 14, \quad \therefore \quad b=-2. \end{aligned}$$

Now, from (1) we have $a+4\times -2=8$, $\therefore \quad a=16$.

$\therefore$ the requd. series $=16, 14, 12, 10, \ldots$

Ex. 7. Which term of the series 2, 5, 8,...is 59 ?

Let 59 be the nth term of the series. Here $a=2$, $b=5-2=3$.

$\therefore \quad a+(n-1)b=59$,

or, $2+(n-1)\times 3=59$, or, $3(n-1)=57$,

or, $n-1=19$, $\therefore \quad n=20$.

$\therefore$ 59 is the 20th term of the series.

Ex. 8. Is 46 a term of the series 1, 4, 7...?

If possible, let 46 be the nth term of the series, Here $a=1$, and $b=4-1=3$.

Now, $a+(n-1)\,b=46$, or, $1+(n-1)\times 3=46$,

or, $3(n-1)=45$, or, $n-1=15$, $\therefore \; n=16$.

Hence 46 is the 16th term of the series.

Ex. 9. If the pth and qth terms of an A. P. are respectively q and p, find the first term and the common difference.

Let the first term $=a$, and the com. diff. $=b$.

$\therefore$ from the given data, $a+(p-1)b=q\ldots(1)$

and $a+(q-1)b=p\ldots(2)$

$\therefore$ (subtracting) $(p-q)b=q-p$. $\therefore$ $b=\dfrac{q-p}{p-q}=-1$.

Now, from (1) we have $a+(p-1)\times-1=q$, or, $a-p+1=q$,

$\therefore$ $a=p+q-1$.

$\therefore$ the first term $=p+q-1$ and the common difference $=-1$.

Ex. 10. The mth term of an A.P. is n and the nth term is m. Find the pth term of it. (C. U. '47 ; E. B. S. B. '51)

Let $t_1=a$, and the com. diff. $=b$.

$\therefore$ here $a+(m-1)b=n\ldots(1)$

and $a+(n-1)b=m\ldots(2)$

$\therefore$ (subtracting) $(m-n)b=n-m$, $\therefore$ $b=\dfrac{n-m}{m-n}=-1$.

Now from (1) we have $a+(m-1)\times(-1)=n$,

or, $a-m+1=n$, $\therefore$ $a=m+n-1$.

$\therefore$ the reqd. pth term $=a+(p-1)b$

$=m+n-1+(p-1)\times(-1)=m+n-1-p+1=m+n-p$.

Ex. 11. If a be the first term and l the last term of an A. P., show that the sum of the pth term from the beginning and the pth term from the end is $a+l$.

Let $t_1=a$, and the common difference $=b$.

$\therefore$ the pth term from the beginning $=a+(p-1)b$

and the pth term from the end $=l-(p-1)b$

$\therefore$ the sum of the two terms $=a+(p-1)b+l-(p-1)b=a+l$.

[**Note :** If any particular term from the beginning is $a+rd$, then the equidistant term (*i.e.* that particular term) from the end is $l-rd$ in any A. P.]

Exercise 2

1. The first term of an A. P. is 6 and the common difference is 2. Find the 15th term. [C. U. '22]

2. Find the common difference of an A. P. of which the 1st term is 1 and the 10th term is 10. [C. U. '25]

3. The first term of an A. P. is 2 and the 20th term is 59. Find the common difference. [C. U. '24]

4. Find the series in A. P. of which the first term and the common difference are a and d respectively.

5. Find the series in A. P. of which the 1st term is 5 and the common difference is -3.

6. Find the 15th term of the series 16, 13, 10,.....

7. Find the 12th and the rth terms of the series 2, 5, 8,...

8. Find the 20th and the nth terms of the series 8, 6, 4,...

9. Find the 10th term of the series $1+\frac{5}{4}+1\frac{1}{2}+\ldots\ldots$

10. Find the nth term of the series (*i*) $a, (a+\frac{1}{n}), (a+\frac{2}{n}),\ldots$ and of (*ii*) $\frac{1}{n}, \frac{n+1}{n}, \frac{2n+1}{n},\ldots\ldots$ [C. U. 1886]

11. Which term of the series 6, 10, 14,......is 38 ?

12. Which terms of the series 10, 8, 6,...are 0 and -10 ?

13. Is 29 a term of the series 3, 6, 9,...?

14. Is $6\frac{3}{5}$ a term of the series $\frac{3}{5}, 1, \frac{7}{5},\ldots$?

15. The second term of an A. P. is 6 and the fourth term is 14. Find the 10th term. [C. U. '29]

16. Find the 20th term of an A. P. of which the 5th and the 12th terms are -4 and -25 respectively.

17. Find the first and the 10th terms of the A. P. whose 5th and 13th terms are 5 and -3 respectively.

18. Find the series in A. P. of which the pth term is $3p-1$.

19. If the pth and qth terms of an A. P. be respectively c and d, find the first term and the common difference. [C. U. '34]

20. If the mth term of an A. P. is n and the nth term is m, find the $(m+n)$th term.

21. If the pth term of an A. P. is a and the qth term is b, find the rth term.

22. Prove that in an A. P. the sum of two equidistant terms from the beginning and the end is constant.

ARITHMETIC MEAN

6. If three quantities are in A. P., the middle term is called the **Arithmetic Mean** between the other two. The arithmetic mean may be written as A. M. Thus, if a, b, c be in A. P., then b is the arithmetic mean between a and c. Similarly 6 is the A. M. between 4 and 8, because 4, 6, 8 are in A. P.

If several terms are in A. P., then the terms between the first and last are called the arithmetic means between them.

Thus, if $a, m_1, m_2, m_3, \ldots, m_n, b$ be in A.P., then $m_1, m_2, \ldots, m_n$ are n arithmetic means between a and b.

Examples (3)

Ex. 1. Find the Arithmetic mean between a and b.

[C. U. '48]

Let m be the arithmetic mean. Then a, m, b are in A. P.

$\therefore\quad m-a=b-m$ (each side being equal to com. diff.)

or, $\quad 2m=a+b, \qquad \therefore\quad m=\frac{a+b}{2}$.

[**N. B.** : The arithmetic mean between two quantities is half their sum.]

Ex. 2. Find n arithmetic means between a and b.

The n means together with a and b form an A. P. of $n+2$ terms of which the first term is a and the $(n+2)$th term is b.

Let d be the common difference.

$$\because\ t_{n+2}=b,\quad \therefore\quad a+(n+2-1)d=b, \text{ or, } (n+1)d=b-a,$$
$$\therefore\quad d=\frac{b-a}{n+1}.$$

Hence the required means are

$$\left(a+\frac{b-a}{n+1}\right),\ \left(a+2.\frac{b-a}{n+1}\right),\ \left(a+3.\frac{b-a}{n+1}\right),\ldots,\left(a+n.\frac{b-a}{n+1}\right).$$

[**N. B.** : The first mean is obtained by adding the common difference to the first term, the 2nd mean is obtained by adding the com. diff. to the first mean, and so on. Here, the last mean is $n.\frac{b-a}{n+1}$, it may also be written as $b-d$ or $b-\frac{b-a}{n+1}$.]

Ex. 3. Insert 10 arithmetic means between 2 and 57. [C. U. '19 ; D. B. '28]

The 10 means together with 2 and 57 at the two ends will form an A. P., whose 1st term is 2 and the 12th term is 57. Let b be the common difference.

$\therefore \; t_{12}=57$, or, $2+11b=57$, or, $11b=55$, $\therefore \; b=5$.

$\therefore$ 1st mean$=2+5=7$, 2nd mean$=7+5=12$, thus the reqd. means$=7, 12, 17, 22, 27, 32, 37, 42, 47, 52$.

Ex. 4. There are n arithmetic means between 4 and 31 such that the second mean : the last mean$=5:14$. Find n.

The n means together with 4 and 31 will form an A. P. of $(n+2)$ terms of which the first term is 4 and the $(n+2)$th term is 31.

Let b be the common difference.

$\because \; t_{n+2}=4+(n+2-1)b=31$,

$\therefore \; (n+1)b=27 \ldots\ldots (1)$

Again, $\because$ b is the com. diff., $\therefore$ the 2nd mean$=4+2b$ and the last mean$=31-b$.

$\therefore$ from the given condition $\frac{4+2b}{31-b}=\frac{5}{14}$,

or, $28b+56=155-5b$, oa, $33b=99$, $\therefore \; b=3$.

Now, from (1) we have $(n+1)\times 3=27$,

or, $n+1=9$, $\therefore \; n=8$.

Exercise 3

Find the arithmetic mean between :—

1. 7 and 23. **2.** -5 and 13. **3.** -4 and -14.

4. $(x-a)^2$ and $(x+a)^2$ **5.** $2\frac{1}{4}$ and $3\frac{1}{2}$. **6.** $\frac{m+n}{m-n}$ and $\frac{m-n}{m+n}$.

7. Insert 7 arithmetic means between 1 and 41. [C. U. '14]

8. Insert 4 arithmetic means between 4 and 324. [C. U. 1890]

9. There are n arithmetic means between 2 and 23 and the first mean : the last mean$=1:4$; find n.

10. There are n arithmetic means between 14 and 38 such that the 2nd mean : the last mean$=4:7$; find n.

7. Sum of a series in A. P. to any number of terms

Show how to find the sum of n terms of an A. P., being given the first term and the commen difference.

[C. U. '30. '33, '37. '41, '43, 44, '45 ; D. B. '34, '38, '43 ; [G. U. '49]

Let a be the first term, b the common difference, n the number of terms of a series in A. P.

Let S be the sum and l the last term.

Then, $S=a+(a+b)+(a+2b)+\dots+(l-2b)+(l-b)+l$

and writing the series in the reverse order,

we get $\underline{S=l+(l-b)+(l-2b)+\dots+(a+2b)+(a+b)+a}$

Now adding the corresponding terms we have

$$2S=(a+l)+(a+l)+(a+l)+\dots+(a+l)+(a+l)+(a+l)$$
$$=(a+l)+(a+l)+\dots\text{to } n \text{ terms}=n(a+l),$$

$\therefore \quad \mathbf{S=\frac{n}{2}(a+}l\mathbf{)}\dots(1).$

Again, $\because$ l is the last or nth term, $\therefore$ $l=a+(n-1)b$.

$\therefore$ From (1) we have $\mathbf{S=\frac{n}{2}\{2a+(n-1)b\}}\dots(2)$.

[**N. B.** Apply the formula-(1) when the first and last terms are known and apply the formula-(2) in other cases. Note that sum to n terms may be denoted by S_n.]

Examples (4)

Ex. 1. Sum to 21 terms the Arithmetic progression $3+7+11+\dots$

Here the first term $a=3$, the (c. d.) $b=7-3=4$ and $n=21$. Let S be the reqd. sum.

$\therefore \quad S=\frac{n}{2}\{2a+(n-1)b\}=\frac{21}{2}\{3\times2+(21-1)\times4\}=\frac{21}{2}\times86=903.$

Ex. 2. The first two terms of an A. P. are 3 and 1. Write down the 10th term and the sum of the first 10 terms.

[C. U. '13]

Here the first term $a=3$, the (c. d.) $b=1-3=-2$, $n=10$.

Let S be the sum.

$\therefore \quad t_{10}=a+(n-1)b=3+(10-1)\times(-2)=3-18=-15$;

and $S=\frac{n}{2}(a+l)=\frac{10}{2}(3-15)=5\times(-12)=-60.$

Ex. 3. Find the sum of $3+5+7+\ldots$to n terms.

Here, $a=3$, $b=5-3=2$. Let S be the sum

$$\therefore \quad S=\tfrac{n}{2}\{2a+(n-1)b\}$$
$$=\tfrac{n}{2}\{2\times 3+(n-1)\times 2\}=\tfrac{n}{2}\{2n+4\}=n^2+2n.$$

Ex. 4. Find the sum of the series $5+7+9+\ldots+65$.

Here, $a=5$, $b=7-5=2$. Let the number of terms be n.

Then the last or the nth term$=65$.

$\therefore \quad a+(n-1)b=65$, or, $5+(n-1)\times 2=65$,

or, $2(n-1)=60$, or, $n-1=30$, $\therefore \quad n=31$.

$\therefore$ the reqd. sum$=\frac{31}{2}(5+65)=\frac{31}{2}\times 70=1085$.

Ex. 5. Find the A. P. of which the sum to n terms is $(2n+1)(2n-1)$.

Let S_n denote the sum of n terms,

$\therefore \quad S_n=(2n+1)(2n-1)=4n^2-1$.

If we put 1, 2, 3, etc. for n, we have S_1 (*i.e.* sum of the first 1 term), S_2 (*i.e.*, the sum of the first 2 terms), and so on.

$\therefore \quad S_1=4.1^2-1=3$, $S_2=4.2^2-1=15$, $S_3=4.3^2-1=35,\ldots$

Again, $t_1=S_1=3$ [$\because$ the sum of the first 1 term and the first term are the same.]

$t_2=S_2-S_1=15-3=12$ [$\because$ we get the 2nd term by subtracting the first term form the sum of the first two terms]

Similarly, $t_3=S_3-S_2=35-15=20$,

and $t_4=S_4-S_3=63-35=28$.

$\therefore$ the required series is 3, 12, 20, 28, ...,here the terms succeeding the first term form an A. P.

[**Note :** In such cases at least the first four terms should be found to determine the series.]

Ex. 6. Find the sum of all the multiples of 13 between 750 and 1000. [C. U. '35]

The multiples of 13 will form an A. P. of which the common difference is 13.

Evidently, the multiple of 13 (*i.e.*, the number divisible by 13) next to 750 will be the first term and the multiple of 13 just preceding 1000 will be the last term of the series.

$$\begin{array}{r|l} 13 & 750 \;(\, 57 \\ & \underline{65} \\ & 100 \\ & \underline{91} \\ & 9 \end{array} \qquad\qquad \begin{array}{r|l} 13 & 1000 \;(\, 76 \\ & \underline{91} \\ & 90 \\ & \underline{78} \\ & 12 \end{array}$$

$$13-9=4$$

$\therefore \quad t_1=750+4=754. \qquad \therefore \quad$ last term $=1000-12=988.$

Let n be the number of terms in the series.

$\therefore$ the nth term, *i e.*, $a+(n-1)b=988$,

or, $754+(n-1)\times 13=988$, or, $13(n-1)=988-754=234$,

or, $n-1=18$, $\therefore \; n=19$.

$\therefore$ the required sum $=\frac{19}{2}(754+988)=\frac{19}{2}\times 1742=16549$.

Ex. 7. Find *without assuming any formula*, the sum of $1+3+5+\ldots\ldots$to 40 terms.

Here $a=1$, the (c. d.) or $b=3-1=2$. Now $t_1=a=1$, $t_2=a+1.b=1+1.2=3$, $t_3=a+2.b=1+2.2=5$,

$\therefore$ evidently $t_{40}=a+39.b=1+39.2=79$. Let S be the sum.

$$\begin{array}{l} \therefore \; S=1+3+5+\ldots+75+77+79 \\ \underline{\text{also } S=79+77+75+\ldots+5+3+1 \text{ (writing in reverse order)}} \\ \therefore \; 2S=80+80+80+\ldots\text{to 40 terms}=80\times 40 \\ \qquad \therefore \; S=\frac{80\times 40}{2}=1600. \end{array}$$

Ex. 8. Find, without assuming any formula, the sum of the first n terms of the series $1+3+5+7+\ldots$

Here $a=1$, the com. diff. $b=3-1=2$,

Now $t_1=1$, $t_2=1+1.2=3$, $t_3=1+2.2=5$, $t_4=1+3.2=7$.

Similarly $t_n=1+(n-1)\,2=2n-1$.

$$\begin{array}{l} \therefore \; s=1+3+5+\ldots+(2n-5)+(2n-3)+(2n-1) \\ \underline{\text{Also, } s=(2n-1)+(2n-3)+(2n-5)+\ldots+5+3+1} \\ \therefore \; \text{(adding) } 2s=2n+2n+2n+\ldots\text{to } n \text{ terms}=2n\times n=2n^2. \\ \therefore \; s=n^2. \end{array}$$

Ex. 9. Find the sum of the series $39+37+35+\ldots+3$, without assuming the formula of summation of a series in A. P.

Here $a=39$, $b=37-39=-2$. Let the number of terms be n.

$\therefore\ a+(n-1)b=3$, or, $39+(n-1)\times(-2)=3$,

or, $-2(n-1)=-36$, or, $n-1=18$, $\therefore\ n=19$.

$\therefore\ s=39+37+35+\ldots\ldots+7+5+3$

also, $s=3+5+7+\ldots\ldots+35+37+39$

$\therefore\ 2s=42+42+42+\ldots\ldots$to 19 terms $[\because\ n=19]$

$=42\times19$, $\quad\therefore\ s=\frac{42\times19}{2}=399$.

[**Note :** In Ex. 7 and Ex. 8, no formula can be applied, but in Ex. 9 only the application of the formula of summation is not permitted, so here the formula for finding n has been applied.]

Ex. 10. Find the sum of $3+4+8+9+13+14+18+19+\ldots$ to 20 terms. [C. U. 1881]

Here the series can be split up into two series in A. P. $(3+8+13+\ldots)$ and $(4+9+14+\ldots)$,

so the number of terms in each $=10$.

$\therefore\ s=(3+8+13+\ldots$to 10 terms$)+(4+9+14+\ldots$to 10 terms$)$

$=\frac{10}{2}\{6+9\times5\}+\frac{10}{2}\{8+9\times5\}=5\times51+5\times53=520$.

Ex. 11. How many terms of the series $3+5+7+\ldots$must be taken in order that the sum may be equal to 624 ?

[C. U. '39 Sup.]

Let 624 be the sum of n terms. Here $a=3$, $b=5-3=2$.

$\because\ \frac{n}{2}\{2a+(n-1)b\}=s$, $\quad\therefore\ \frac{n}{2}\{2\times3+(n-1)\times2\}=624$,

or, $\frac{n}{2}(2n+4)=624$, or, $n^2+2n=624$,

or, $n^2+2n-624=0$, or, $(n+26)(n-24)=0$, $\therefore\ n=-26$ or 24.

$\because$ the number of terms cannot be negative,

$\therefore$ the reqd. number of terms $=24$.

[**N. B. :** If any three of quantities a, b, n and S in the formula $S=\frac{n}{2}\{2a+(n-1)b\}$ are known, the remaining quantity can be found from the formula.]

Ex. 12. Find the number of terms of the series 17, 5, $-7, \ldots$ whose sum is -78. [D. B. '31]

Let -78 be the sum of n terms of the series.

Here $a=17$, (the com. diff.) $b=5-17=-12$.

$\because \frac{n}{2}\{2a+(n-1)b\}=s, \quad \therefore \frac{n}{2}\{34+(n-1)\times(-12)\}=-78,$

or, $\frac{n}{2}(46-12n)=-78$, or, $23n-6n^2+78=0$,

or, $6n^2-23n-78=0$, or, $(6n+13)(n-6)=0$, $\therefore n=-\frac{13}{6}$ or 6.

$\because$ the number of terms cannot be a fraction or a negative integer, $\therefore$ the reqd. number of terms $=6$.

Ex. 13. The first term of an A. P. is 9 and the last term is 96. If the sum be 1575, find the common difference. [D. B. '32]

Let b be the common difference.

$\because \mathrm{S}=\frac{n}{2}(a+l)$, $\therefore \frac{n}{2}(9+96)=1575$, or, $\frac{n}{2}=\frac{1575}{105}=15$, $\therefore n=30$.

Again, here the last or nth term $=96$, $\therefore a+(n-1)b=96$,

or, $9+29b=96$, or, $29b=87$, $\therefore b=3$.

$\therefore$ the reqd. common difference $=3$.

Ex. 14. The sum of n terms of the series $10+8+6+\ldots$ is 28; find n and explain the double answer.

Here the first term $a=10$, the com. diff. $b=8-10=-2$.

$\because \frac{n}{2}\{2a+(n-1)b\}=s, \quad \therefore \frac{n}{2}\{20+(n-1)\times -2\}=28$

or, $11n-n^2-28=0$, or, $n^2-11n+28=0$, or, $(n-4)(n-7)=0$,

$\therefore n=4$ or 7.

Here both the values of n are admissible, because here the sum of the terms from the 5th to the 7th is 0. The first 7 terms of the series are 10, 8, 6, 4, $\underline{2,\ 0,\ -2,}$ so the sum of the first 4 terms is equal to that of the first 7 terms.

Ex. 15. Show that the sum of n terms of the series 4, 12, 20, 28,...is the square of an even number. [C. U. '27, '39]

Here $a=4$, $b=12-4=8$.

$\therefore \mathrm{S}_n=\frac{n}{2}\{2.4+(n-1)\times 8\}=\frac{n}{2}(8+8n-8)=4n^2=(2n)^2$.

Now, since $2n$ is an even number whether n is odd or even, $\therefore$ the sum of n terms of the series is the square of an even number.

Ex. 16. The sum of n terms of an A. P. is 40, the common difference is 2, and the last term is 13. Find n.

[C. U. '46 ; Pat. U. '18]

Here the nth term is the last term.

$\therefore\ a+(n-1)b=13$, or, $a+(n-1)\times 2=13$,

$\therefore\ a=13-2n+2=15-2n$.

Again, $\because\ \frac{n}{2}(a+l)=\text{S}$,

$\therefore\ \frac{n}{2}(15-2n+13)=40$ [putting the values of a, l, S],

or, $-n^2+14n-40=0$, or, $n^2-14n+40=0$,

or, $(n-4)(n-10)=0$, $\therefore\ n=4$, or, 10.

[**N. B.** Here $a=15-2n$. $\therefore$ if $n=4$, the first term is $(15-2\times 4)$ or 7 ; and if $n=10$, the first term $=15-2\times 10=-5$. In the latter case the sum of the first 6 terms is zero. Hence both the values of n are admissible here.]

Ex. 17. Find the sum of 21 consecutive odd numbers of which the last term is 51.

Here, the last term is 51. $\because$ the terms are consecutive odd numbers, $\therefore$ the common difference $=2$.

Now, writing the series in reverse order we get the series $51+49+47+\ldots$and its com. diff. is -2.

$\therefore$ the reqd. sum $=\frac{21}{2}\{2\times 51+(21-1)\times -2\}$

$=\frac{21}{2}(102-40)=\frac{21}{2}\times 62=651$.

Ex. 18. If the 18th term of an A. P. is 39, find the sum of 35 terms of the series.

Let a be the first term and b the com. diff.

$\therefore\ t_{18}=a+(18-1)b=a+17b$, $\therefore\ a+17b=39$ (given) ;

Now, $\text{S}_{35}=\frac{35}{2}\{2a+(35-1)b\}=\frac{35}{2}\{2a+34b\}$

$=35(a+17b)=35\times 39=1365$.

Ex. 19. The sum of 10 terms of an A. P. is 120 and the sum of 15 terms is 255 ; find the sum of n terms.

Let a be the 1st term and b the common difference.

$\because\ \text{S}_{10}=120$, $\therefore\ \frac{10}{2}\{2a+(10-1)b\}=120$,

or, $5(2a+9b)=120$, $\therefore\ 2a+9b=24$......(1)

Again, $\because$ $S_{15}=255$, $\therefore$ $\frac{15}{2}(2a+14b)=255$, or, $2a+14b=34$...(2)

Solving (1) and (2) we have $a=3$, $b=2$.

$\therefore$ $S_n=\frac{n}{2}\{2.3+(n-1)\times 2\}=\frac{n}{2}(2n+4)=n(n+2)$.

Ex. 20. A class consists of a number of boys whose ages are in A. P., the common difference being four months. If the youngest boy is just 8 years old, and if the sum of the ages is 168 years, find the number of boys in the class. [N.B. '64, C.U. 1872]

Let n be the number of boys.

Here the first term $=8$ years, and tbe c. d. $=4$ months $=\frac{1}{3}$ yr.

$\therefore$ $\frac{n}{2}\{2\times 8+(n-1)\times\frac{1}{3}\}=168$, or, $\frac{n}{2}(16+\frac{n-1}{3})=168$,

$\therefore$ $\frac{n}{2}\times\frac{n+47}{3}=168$, or, $n^2+47n=1008$,

or, $n^2+47n-1008=0$, or, $(n-16)(n+63)=0$,

$\therefore$ $n=16$, or, -63.

$\because$ the number of boys cannot be negative,

$\therefore$ the reqd. number of boys $=16$.

Ex. 21. In boring a well 200 ft. deep the cost is Re. 1. 2 as. for the first foot and an additional anna for each subsequent foot. What is the cost of boring the well ? [C. U. 1934]

From the given conditions we have

$a=18$ as, $b=1$ a. and $n=200$.

$\therefore$ the required cost (i.e., S) $=\frac{200}{2}\{2.18+(200-1)\times 1\}$ as.

$=100\times 235$ as. $=23500$ as. $=$ Rs. 1468.12 as.

Ex. 22. Two travellers start together on the same road. One of them travels uniformly 10 miles a day. The other travels 8 miles the first day and increases his pace by half a mile a day each succeeding day. After how many days will the latter overtake the former ? [Pat. U. '20]

Let n be the number of days. The two men will travel the same distance in these n days.

Now, $\because$ the first man travels 10 miles each day,

$\therefore$ in n days he travels $10n$ miles...(1).

The second traveller goes 8 miles on the 1st day, $8\frac{1}{2}$ miles on the 2nd day, 9 miles on the 3rd, and so on.

$\therefore$ Here the first term $a=8$, and the common difference $b=\frac{1}{2}$.

$\therefore$ in n days he travels $\frac{n}{2}\{2.8+(n-1)\times\frac{1}{2}\}$ miles

or, $\frac{n}{2}(\frac{31+n}{2})$ miles......(2).

$\therefore$ From (1) and (2) we have $\frac{n}{2}(\frac{31+n}{2})=10n$,

or, $\frac{31+n}{2}=20$, or, $31+n=40$, $\therefore$ $n=9$.

Hence the required number of days $=9$.

8. Sum of the Natural numbers

The numbers 1, 2, 3, 4,...etc. are called natural numbers. The first n natural numbers are the numbers 1, 2, 3,......,n.

I. Find the sum of the first n natural numbers.

Here the first term $a=1$, com. diff. $b=1$ and the number of terms $=n$. Let S be the sum.

$$\therefore \quad S=1+2+3+\ldots+n=\frac{n}{2}(1+n)=\mathbf{\frac{n(n+1)}{2}}.$$

II. Find the sum of the squares of the first n natural numbers.

[C. U. '15 ; D. B. '32, '34, '45 ; G. U. '51]

Let s be the sum, then $s=1^2+2^2+3^2+\ldots\ldots+n^2$.

Now, $n^3-(n-1)^3=3n^2-3n+1$ [Identically, *i.e.*, the two sides are equal for any value of n].

Putting 1, 2, 3,..., n for n in the identity we have,

$$1^3-0^3=3.1^2-3.1+1$$
$$2^3-1^3=3.2^2-3.2+1$$
$$3^3-2^3=3.3^2-3.3+1$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$n^3-(n-1)^3=3n^2-3n+1$$

$$(\text{Adding}) \quad n^3=3(1^2+2^2+3^2+\ldots+n^2) - 3(1+2+3+\ldots+n)+n.$$

$$\text{or,}\quad n^3 = 3s - \frac{3n(n+1)}{2} + n,$$

$$\text{or,}\quad 3s = n^3 + \frac{3n(n+1)}{2} - n = \frac{2n^3 + 3n(n+1) - 2n}{2}$$

$$= \frac{n(2n^2+3n+1)}{2} = \frac{n(n+1)(2n+1)}{2}$$

$$\therefore\quad \mathbf{s} = \frac{\mathbf{n(n+1)(2n+1)}}{\mathbf{6}}.$$

[**N. B.** On adding the terms on the L.H.S. the sum is n^3, as all other terms cancel one another. On the R. H. S. 3 is written as the common factor. Adding $1+1+1+\ldots$to n terms we get $1\times n$, *i.e.*, n.]

III. *Find the sum of the cubes of the first* **n** *natural numbers.* [C. U. '18]

Let s be the sum. Then $s = 1^3+2^3+3^3+\ldots+n^3$.

Now, $n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1$ [Identically]

Putting 1, 2, 3,…,n for n in the above identity we have

$$\begin{array}{l}1^4 - 0^4 = 4.1^3 - 6.1^2 + 4.1 - 1\\ 2^4 - 1^4 = 4.2^3 - 6.2^2 + 4.2 - 1\\ 3^4 - 2^4 = 4.3^3 - 6.3^2 + 4.3 - 1\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \underline{n^4 - (n-1)^4 = 4.n^3 - 6.n^2 + 4.n - 1}\end{array}$$

$$(\text{ adding })\ n^4 = 4(1^3+2^3+3^3+\ldots+n^3)$$
$$-6(1^2+2^2+3^2+\ldots+n^2) + 4(1+2+3+\ldots+n) - n,$$

$$\text{or,}\quad n^4 = 4s - \frac{6n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} - n,$$

$$\therefore\quad 4s = n^4 + n(n+1)(2n+1) - 2n(n+1) + n$$
$$= n^4 + n + n(n+1)(2n+1) - 2n(n+1)$$
$$= n(n+1)(n^2-n+1) + n(n+1)(2n+1) - 2n(n+1)$$
$$= n(n+1)(n^2 - n + 1 + 2n + 1 - 2)$$
$$= n(n+1)(n^2+n) = n(n+1)n(n+1) = n^2(n+1)^2$$

$$\therefore\quad s = \frac{n^2(n+1)^2}{4} = \left\{\frac{n(n+1)}{2}\right\}^2.$$

[**N. B.** From I and III we find that the sum of III is the square of the sum of I, *i.e.*,

$$1^3+2^3+3^3+\ldots+n^3=(1+2+3+\ldots+n)^2]$$

The application of these important formulas is illustrated below.

Examples (5)

Ex. 1. Sum to n terms the series whose nth term is $n(n-1)$.

Let s be the sum. Here $t_n=n(n-1)=n^2-n$.

Now, putting 1, 2, 3,......, n for n we get

$$\begin{aligned}t_1&=1^2-1\\t_2&=2^2-2\\t_3&=3^2-3\\&\ldots\ldots\ldots\ldots\\t_n&=n^2-n\end{aligned}$$

(adding) $s=(1^2+2^2+3^2+\ldots+n^2)-(1+2+3+\ldots+n)$

$$=\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}=\frac{n(n+1)(2n+1)-3n(n+1)}{6}$$

$$=\frac{n(n+1)(2n-2)}{6}=\frac{2n(n+1)(n-1)}{6}=\frac{n(n^2-1)}{3}.$$

[**N.B.** In adding the L. H. S. we put s for $t_1+t_2+\ldots\ldots+t_n$.]

Ex. 2. Sum to n terms the series $1.2+2.3+3.4+\ldots\ldots$

[C. U. '12, '17, '39 Sup., '44, '51 ; D. B. '26, '33, '41, '44]

Here each term consists of two factors. So the nth term also consists of two factors.

Now, the first factors of the terms of the series form the A.P. 1, 2, 3,...and the second factors form the A. P. 2, 3, 4,...

$\therefore$ here $t_n=(n\text{th term } 1, 2, 3,\ldots)\times(t_n \text{ of } 2, 3, 4,\ldots)$
$=n.(n+1)=n^2+n$. Now, putting 1, 2, 3,..., n for n we get

$$\begin{aligned} t_1&=1^2+1\\ t_2&=2^2+2\\ t_3&=3^2+3\\ &\ldots\ldots\ldots\ldots\\ t_n&=n^2+n \end{aligned}$$

$$\therefore\quad s=(1^2+2^2+3^2+\ldots+n^2)+(1+2+3+\ldots+n)$$

$$=\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}=\frac{n(n+1)(2n+1)+3n(n+1)}{6}$$

$$=\frac{n(n+1)(2n+4)}{6}=\frac{2n(n+1)(n+2)}{6}=\frac{n(n+1)(n+2)}{3}.$$

[**N. B.** In all such summations we have to find the nth term (t_n) first and then to put 1, 2, 3,$\cdots$for n. By inspection we have to adopt a method for finding the t_n of a series. In cases where each term is the product of 2 or more factors, we adopt the method shown above.]

Ex. 3. Sum the series $1^2+3^2+5^2+\ldots$to n terms. [C.U. '59]

Here $t_n=$the square of the nth term of $1+3+5+\ldots\ldots$

$$=\{1+(n-1)\times2\}^2=(2n-1)^2=4n^2-4n+1.$$

Now, putting 1, 2, 3,..., n for n we have

$$\begin{aligned} t_1&=4.1^2-4.1+1\\ t_2&=4.2^2-4.2+1\\ t_3&=4.3^2-4.3+1\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ t_n&=4n^2-4.n+1 \end{aligned}$$

(addidg) $s=4(1^2+2^2+3^2+\ldots+n^2)-4(1+2+\ldots+n)+n$

$$=\frac{4n(n+1)(2n+1)}{6}-\frac{4n(n+1)}{2}+n$$

$$=\frac{2n(n+1)(2n+1)-6n(n+1)+3n}{3}=\frac{n(4n^2-1)}{3}.$$

Ex. 4. Sum to n terms $2^2+5^2+8^2+\ldots\ldots$ [D. B. '35]

Here t_n = the square of the nth term of $2+6+8+\ldots\ldots$

$$=\{2+(n-1)\times 3\}^2=(3n-1)^2=9n^2-6n+1.$$

Putting 1, 2, 3,..., n for n we have

$$t_1=9.1^2-6.1+1$$

$$t_2=9.2^2-6.2+1$$

$$t_3=9.3^2-6.3+1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$t_n=9.n^2-6n+1$$

(Adding) $s=9(1^2+2^2+\ldots+n^2)-6(1+2+\ldots+n)+n$

$$=\frac{9n(n+1)(2n+1)}{6}-\frac{6n(n+1)}{2}+n$$

$$=\frac{3n(n+1)(2n+1)-6n(n+1)+2n}{2}=\frac{n(6n^2+3n-1)}{2}.$$

Ex. 5. Sum to n terms $1+4+8+13+19+26+\ldots\ldots$

Let s be the sum and t_n denote the nth term.

Then, $s=1+4+8+13+19+26+\ldots\ldots+t_n$

also $s=\quad 1+4+\ 8+13+19+\ldots\ldots+t_{n-1}+t_n$

(Subtracting) $0=1+3+4+5+6+7+\ldots\ldots+(t_n-t_{n-1})-t_n$

$\therefore\quad t_n=1+3+4+5+6+\ldots$to n terms [transposing]

$$=1+\{3+4+5+6+\ldots+\text{to }(n-1)\text{ terms}\}$$

$$=1+\frac{n-1}{2}\{6+(n-2)\times 1\}=\frac{n^2+3n-2}{2}=\tfrac{1}{2}n^2+\tfrac{3}{2}n-1.$$

Now, putting 1, 2, 3,..., n for n we have

$$t_1=\tfrac{1}{2}.1^2+\tfrac{3}{2}.1-1$$

$$t_2=\tfrac{1}{2}.2^2+\tfrac{3}{2}.2-1$$

$$t_3=\tfrac{1}{2}.3^2+\tfrac{3}{2}.3-1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$t_n=\tfrac{1}{2}n^2+\tfrac{3}{2}n-1$$

(Adding) $s=\frac{1}{2}(1^2+2^2+\ldots+n^2)+\frac{3}{2}(1+2+\ldots+n)-n$

$$=\frac{n(n+1)(2n+1)}{12}+\frac{3n(n+1)}{4}-n$$

$$=\frac{n(n+1)(2n+1)+9n(n+1)-12n}{12}$$

$$=\frac{2n(n^2+6n-1)}{12}=\frac{n(n^2+6n-1)}{6}.$$

[**N. B.** Here t_n is found by a different method. Here, the terms of the given series are not in A. P., but the successive differences of the terms form an A. P. Hence the differences are found first. Here, note that the differences excepting the first one are in A.P. Hence the number of terms of the A.P. series formed by them is $(n-1)$ and not n.]

Ex. 6. Sum to n terms the series $1.2.3+2.3.4+3.4.5+\ldots$

Here $t_n=(t_n \text{ of } 1, 2, 3,\ldots)\times(t_n \text{ of } 2, 3, 4,\ldots)\times(t_n \text{ of } 3, 4, 5\ldots)$

$$=n(n+1)(n+2)=n^3+3n^2+2n.$$

Now, putting 1, 2, 3,..., n for n we get

$$t_1=1^3+3.1^2+2.1$$
$$t_2=2^3+3.2^2+2.2$$
$$t_3=3^3+3.3^2+2.3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$t_n=n^3+3.n^2+2.n$$

$\therefore$ (Adding) $s=(1^3+2^3+\ldots+n^3)+3(1^2+2^2+\ldots+n^2)+2(1+2+\ldots+n)$

$$=\left\{\frac{n(n+1)}{4}\right\}^2+\frac{3n(n+1)(2n+1)}{6}+\frac{2n(n+1)}{2}$$

$$=\frac{n^2(n+1)^2}{4}+\frac{n(n+1)(2n+1)}{2}+n(n+1)$$

$$=n(n+1)\left\{\frac{n(n+1)}{4}+\frac{2n+1}{2}+1\right\}=n(n+1)\left\{\frac{n^2+5n+6}{4}\right\}$$

$$=\frac{n(n+1)(n+2)(n+3)}{4}.$$

Ex. 7. Sum to n terms the series $1.3^2+2.4^2+3.5^2+\ldots$

Here, $t_n=(t_n \text{ of } 1, 2, 3,\ldots)\times(\text{the square of } t_n \text{ of } 3, 4, 5,\ldots)$

$$=n\times(n+2)^2=n^3+4n^2+4n.$$

$\therefore$ Putting $1, 2, 3,\ldots, n$ for n we have

$$\begin{aligned}
t_1&=1^3+4.1^2+4.1\\
t_2&=2^3+4.2^2+4.2\\
t_3&=3^3+4.3^2+4.3\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
t_n&=n^3+4.n^2+4.n
\end{aligned}$$

$$\therefore\quad s=(1^3+2^3+\ldots+n^3)+4(1^2+2^2+\ldots+n^2)+4(1+2+\ldots+n)$$

$$=\left\{\frac{n(n+1)}{2}\right\}^2+\frac{4n(n+1)(2n+1)}{6}+\frac{4n(n+1)}{2}$$

$$=\frac{n^2(n+1)^2}{4}+\frac{2n(n+1)(2n+1)}{3}+2n(n+1)$$

$$=n(n+1)\left\{\frac{n(n+1)}{4}+\frac{2(2n+1)}{3}+2\right\}=\frac{n}{12}(n+1)(3n^2+19n+32).$$

Ex. 8. Sum to n terms $(1)+(1+2)+(1+2+3)+\ldots$

Here evidently $t_n=(1+2+3+\ldots+n)=\dfrac{n(n+1)}{2}=\frac{1}{2}n^2+\frac{1}{2}n.$

Now, putting $n=1, 2, 3,\ldots\ldots,n$ we have

$$\begin{aligned}
t_1&=\tfrac{1}{2}.1^2+\tfrac{1}{2}.1\\
t_2&=\tfrac{1}{2}.2^2+\tfrac{1}{2}.2\\
t_3&=\tfrac{1}{2}.3^2+\tfrac{1}{2}.3\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
t_n&=\tfrac{1}{2}n^2+\tfrac{1}{2}n
\end{aligned}$$

$$\therefore\quad (\text{adding})\ s=\tfrac{1}{2}(1^2+2^2+\ldots+n^2)+\tfrac{1}{2}(1+2+\ldots+n)$$

$$=\frac{n(n+1)(2n+1)}{12}+\frac{n(n+1)}{4}=n(n+1)\left(\frac{2n+1}{12}+\frac{1}{4}\right)$$

$$=\frac{n(n+1)(n+2)}{6}.$$

Ex. 9. Sum to n terms $(1)+(2+3)+(4+5+6)+\ldots$

Here we get the natural numbers 1, 2, 3, 4, 5,... by removing the brackets. Now we are to determine how many natural numbers can be had from the n terms (*i. e.*, the n brackets) of the given series. We find that the number of terms in the first bracket is 1, that in the second bracket is 2, that in the 3rd is 3 and so on. Hence the total number of terms obtained from the n brackets $=1+2+3+\ldots+n=\dfrac{n(n+1)}{2}$.

$$\therefore \text{ the reqd. sum}=1+2+3+\ldots\ldots\text{to } \frac{n(n+1)}{2} \text{ terms}$$

$$=\frac{1}{2}\left\{\frac{n(n+1)}{2}\right\}^2+\frac{1}{2}\left\{\frac{n(n+1)}{2}\right\} \quad \left[\because \; S_n=\frac{n(n+1)}{2}=\tfrac{1}{2}n^2+\tfrac{1}{2}n, \; \therefore \text{ here } \frac{n(n+1)}{2} \text{ is put for } n\right]$$

$$=\frac{1}{2}\cdot\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2}+1\right\}=\frac{n(n+1)}{4}\times\frac{n^2+n+2}{2}$$

$$=\frac{n(n+1)(n^2+n+2)}{8}.$$

Ex. 10. Sum $1-2+3-4+5-6+\ldots$ to n terms.

(i) If n is an even number, then

the series $=(1-2)+(3-4)+(5-6)+\ldots$ to $\frac{n}{2}$ terms

$=(-1)+(-1)+(-1)+\ldots$ to $\frac{n}{2}$ terms

$\therefore$ the sum $=-1\times\frac{n}{2}=-\frac{n}{2}$.

(ii) If n is an odd number, then

the series $=1+\{(-2+3)+(-4+5)+(-6+7)+\ldots$ to $\dfrac{n-1}{2}$ terms*$\}$

$=1+\{1+1+1\ldots$ to $\dfrac{n-1}{2}$ terms $\}$

$\therefore$ the sum $=1+1\times\dfrac{n-1}{2}=1+\dfrac{n-1}{2}=\dfrac{n+1}{2}$.

[*Here, as n (the number of terms) is odd, $n-1$ will be even. Now, if we put each pair of terms within brackets beginning from the 2nd term, we get $\dfrac{n-1}{2}$ such brackets or terms.]

Ex. 11. Sum to n terms $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots\ldots$

Here $t_n=\frac{1}{(t_n \text{ of } 1, 2, 3,\ldots)\times(t_n \text{ of } 2, 3, 4,\ldots)}=\frac{1}{n(n+1)}$.

$$\text{Now, } t_1=\frac{1}{1.2}=1-\frac{1}{2}$$

$$t_2=\frac{1}{2.3}=\frac{1}{2}-\frac{1}{3}$$

$$t_3=\frac{1}{3.4}=\frac{1}{3}-\frac{1}{4}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$t_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$$

Adding $s=1-\frac{1}{n+1}=\frac{n+1-1}{n+1}=\frac{n}{n+1}$.

Ex. 12. Sum to n terms $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots$ [W. B. S. F. '53]

Here $t_n=\frac{1}{(t_n \text{ of } 1, 3, 5,\ldots)(t_n \text{ of } 3, 5, 7,\ldots)}$

$$=\frac{1}{(2n-1)(2n+1)}.$$

$$\text{Now, } t_1=\frac{1}{1.3}=\frac{1}{2}\left(1-\frac{1}{3}\right)$$

$$t_2=\frac{1}{3.5}=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)$$

$$t_3=\frac{1}{5.7}=\frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$t_n=\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$$

(Adding) $s=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)=\frac{1}{2}\times\frac{2n+1-1}{2n+1}=\frac{n}{2n+1}$.

[**N. B.** Here $1-\frac{1}{3}=\frac{2}{3}$, $\therefore$ $\frac{1}{1.3}$ can be written as $\frac{1}{3}=\frac{1}{2}(1-\frac{1}{3})$.

Note that the sum of the given series to infinity $=\frac{1}{2}\times1=\frac{1}{2}$.]

Miscellaneous Examples (6)

Ex. 1. The sum of 3 numbers in A. P. is 27 and their product is 693 ; find the numbers.

Let the numbers be $a-b$, a, $a+b$.

Then from the given conditions $a-b+a+a+b=27\ldots(1)$

and $(a-b).\,a.\,(a+b)=693\ldots(2)$

Now from (1) we have $3a=27$, $\therefore\ a=9$.

From (2) we have $a(a^2-b^2)=693$, or, $9(81-b^2)=693$,

or, $81-b^2=77$, or, $b^2=4$, $\therefore\ b=\pm2$.

$\therefore$ When $b=2$, the terms are $(9-2)$, 9 and $(9+2)$,

i. e., 7, 9, 11 ;

When $b=-2$, the terms are $9-(-2)$, 9, $(9-2)$, *i.e.*, 11, 9, 7.

$\therefore$ The reqd. numbers are 7, 9, 11 ; or, 11, 9, 7.

Ex. 2. Four numbers are in A. P. The sum of the extremes is 10, and the product of the means is 24. Find the numbers.

[C. U. '43]

Let the numbers be $a-3b$, $a-b$, $a+b$, $a+3b$ [c.d. being $2b$]

Then from the given conditions we have

$a-3b+a+3b=10\ldots(1)$, and $(a-b)(a+b)=24\cdots(2)$.

Now from (1), $2a=10$, $\therefore\ a=5$. From (2), $a^2-b^2=24$,

or, $25-b^2=24$, or, $b^2=1$, $\therefore\ b=\pm1$.

$\therefore$ the reqd. numbers $=(5-3)$, $(5-1)$, $(5+1)$ and $(5+3)$,

or, $(5+3)$, $(5+1)$, $(5-1)$ and $(5-3)$

$=2, 4, 6, 8$; or 8, 6, 4, 2.

[**N. B.** (1) If the number of terms be odd, take a as the middle term and b as the common difference. (2) If the number of terms be even, take $a-b$, $a+b$ as the two middle terms and $2b$ as the common difference.]

Ex. 3. Divide 69 into three parts which are in A. P. and such that the product of the first two is 483.

Let the parts be $a-b$, a, $a+b$.

Then $a-b+a+a+b=69\ldots(1)$ and $(a-b)a=483\ldots(2)$.

From (1) we get $3a=69$, $\therefore$ $a=23$.
From (2) we get $(23-b)\times 23=483$, or, $23-b=21$, $\therefore$ $b=2$.
$\therefore$ The reqd. parts $=21, 23, 25$.

Ex. 4. If a, b, c are in A. P., show that $ab+bc=2b^2$.

$\because$ a, b, c are in A. P.,

$\therefore$ $b-a=c-b$, $\therefore$ $2b=a+c$.

Now, multiplying both sides by b we have $2b^2=ab+bc$.

Ex. 5. If a, b, c are in A. P., prove that $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are also in A. P.

$\because$ a, b, c are in A. P.,

$\frac{a}{abc}, \frac{b}{abc}, \frac{c}{abc}$ are also in A. P.

i.e., $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A. P.

[**Note**. If the same number be added to or subtracted from the terms in A. P., or if the terms be multiplied or divided by any number, the results are in A. P.]

Ex. 6. If a^2, b^2, c^2 be in A. P., prove that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are also in A. P. [C.U. '10, '38 ; D. B. '45 ; G. U. '50]

[**Other method**] $\because$ a^2, b^2, c^2 are in A.P.,

$\therefore$ $b^2-a^2=c^2-b^2$.

Now, $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ will be in A. P.,

if $$\frac{1}{c+a}-\frac{1}{b+c}=\frac{1}{a+b}-\frac{1}{c+a}$$

or, if $\frac{b+c-c-a}{(c+a)(b+c)}=\frac{c+a-a-b}{(a+b)(c+a)}$, or, if $\frac{b-a}{b+c}=\frac{c-b}{a+b}$

i.e., if $b^2-a^2=c^2-b^2$; which is true from the given condition.

Hence $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A. P.

Ex. 7. The sum of n terms of a series in A. P. is $5n^2+7n$. Find the first two terms of the series. [C. U. '41]

Let S_n denote the sum of n terms of the series.

Then, $S_n=5n^2+7n$.

Now, putting 1 and 2 for n we have

$$S_1=5.1^2+7.1=12, \text{ and } S_2=5.2^2+7.2=34$$

$\therefore\ t_1=S_1=12$ ($\because$ the sum of the first term is the first term itself)

and $t_2=S_2-S_1=34-12=22$ ($\because$ the difference of the sum of the first term form the sum of the first two terms = the 2nd term).

$\therefore$ The first two terms $=12, 22$.

Ex. 8. The sum of p terms of an A. P. is q and the sum of q terms is p ; find the sum of $p+q$ terms. [C. U. '50]

Let a be the first term and b the com. diff.

Then from the given conditions we have

$$\tfrac{p}{2}\{2a+(p-1)b\}=q\ldots(1)$$

and $$\tfrac{q}{2}\{2a+(q-1)b\}=p\ldots(2)$$

From (1) we have $2ap+p^2b-pb=2q\ldots(3)$

and from (2) we have $2aq+q^2b-qb=2p\ldots(4)$.

Subtracting (4) from (3) we have

$$2a(p-q)+b(p^2-q^2)-b(p-q)=2(q-p)=-2(p-q),$$

$\therefore\ 2a+(p+q)b-b=-2$ [Dividing by $p-q$]

or, $2a+(p+q-1)b=-2\ldots(5)$

$\therefore$ The sum of $p+q$ terms $=\frac{p+q}{2}\{2a+(p+q-1)b\}$

$$=\tfrac{p+q}{2}\times(-2) \text{ [from (5)]} =-(p+q).$$

Ex. 9. Show that the sum of the latter half of $2n$ terms of any series in A. P. is equal to one-third the sum of $3n$ terms of the same series. [C. U. 1876]

Let a be the first term and b the com. diff.

$\therefore\ S_n=\frac{n}{2}\{2a+(n-1)b\}\ldots(1)$,

$S_{2n}=\frac{2n}{2}\{2a+(2n-1)b\}=n\{2a+(2n-1)b\}\ldots(2)$

and $S_{3n}=\frac{3n}{2}\{2a+(3n-1)b\}\ldots(3)$

Now, the sum of the latter half of $2n$ terms $= S_{2n} - S_n$

$$= n\{2a+(2n-1)b\} - \tfrac{n}{2}\{2a+(n-1)b\}$$
$$= \tfrac{n}{2}\{4a+4bn-2b-2a-bn+b\}$$
$$= \tfrac{n}{2}\{2a+3bn-b\} = \tfrac{n}{2}\{2a+(3n-1)b\}$$

Again, $\frac{1}{3} \times S_{3n} = \frac{1}{3} \times \frac{3n}{2}\{2a+(3n-1)b\} = \frac{n}{2}\{2a+(3n-1)b\}$.

Hence, the problem is proved.

Ex. 10. The pth term of an A. P. is a and the qth term is b. Show that the sum of the first $p+q$ terms is $\frac{p+q}{2}\{a+b+\frac{a-b}{p-q}\}$.

[M. U. 1887]

Let f be the 1st term and d the com. diff.

then $f+(p-1)d=a$...(1)

and $f+(q-1)d=b$...(2)

$\therefore$ (Subtracting), $d(p-q)=a-b$, $\therefore$ $d=\frac{a-b}{p-q}$.

Again, from (1)+(2) we get $a+b=2f+(p+q-2)d$.

$\therefore$ The sum of $p+q$ terms $=\frac{p+q}{2}\{2f+(p+q-1)d\}$

$$=\frac{p+q}{2}\{2f+(p+q-2)d+d\}=\frac{p+q}{2}\left\{a+b+\frac{a-b}{p-q}\right\}.$$

[**N. B.** Here a and b cannot be taken as t_1 and com. diff. as they are the pth and qth terms in the given sum. So f is taken as the 1st term and d as the com. diff.]

Ex. 11. If a, b and c be respectively the pth, qth and rth terms of an A. P., prove that $a(q-r)+b(r-p)+c(p-q)=0$.

[C. U. '32, '37, '46, S. F. '63]

Let f and d be respectively the first term and the com. diff.

Then from the data, $f+(p-1)d=a$...(1),

$f+(q-1)d=b$...(2), and $f+(r-1)d=c$...(3),

From (1) − (2) we have $(p-q)d=a-b$...(4)

and from (2) − (3) „ „ $(q-r)d=b-c$...(5)

Dividing (4) by (5) we have $\frac{p-q}{q-r}=\frac{a-b}{b-c}$,

$\therefore$ $a(q-r)-b(q-r)=b(p-q)-c(p-q)$,

or, $a(q-r)+b(r-q)-b(p-q)+c(p-q)=0$,
or, $a(q-r)+b(r-q-p+q)+c(p-q)=0$,
$\therefore$ $a(q-r)+b(r-p)+c(p-q)=0$.

Ex. 12. If a, b, c be respectively the sums of p, q and r terms of an A. P., prove that $\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$.
[C. U. '45 ; D. B. '43, '45 ; G. U. '49, '51]

Let f be the 1st term and d the common difference.

Then from the given conditions we have

$\frac{p}{2}\{2f+(p-1)d\}=a\ldots(1)$,
$\frac{q}{2}\{2f+(q-1)d\}=b\ldots(2)$, and $\frac{r}{2}\{2f+(r-1)d\}=c\ldots(3)$.

Dividing (1) by $\frac{p}{2}$ we get $2f+(p-1)d=\frac{2a}{p}\ldots(4)$

Similarly from (2) we have $2f+(q-1)d=\frac{2b}{q}\ldots(5)$

and from (3) we have $2f+(r-1)d=\frac{2c}{r}\ldots(6)$

Now, from (4) – (5), we get $(p-q)d=\frac{2a}{p}-\frac{2b}{q}=2(\frac{a}{p}-\frac{b}{q})\ldots(7)$
and from (5) – (6) „ „ $(q-r)d=2(\frac{b}{q}-\frac{c}{r})\ldots(8)$

$\therefore$ From (7)÷(8) we have $\frac{p-q}{q-r}=\dfrac{\frac{a}{p}-\frac{b}{q}}{\frac{b}{q}-\frac{c}{r}}$

$\therefore$ $\frac{a}{p}(q-r)-\frac{b}{q}(q-r)=\frac{b}{q}(p-q)-\frac{c}{r}(p-q)$ [by cross mult.]
or, $\frac{a}{p}(q-r)-\frac{b}{q}(q-r)-\frac{b}{q}(p-q)+\frac{c}{r}(p-q)=0$,
or, $\frac{a}{p}(q-r)-\frac{b}{q}(q-r+p-q)+\frac{c}{r}(p-q)=0$,
or, $\frac{a}{p}(q-r)-\frac{b}{q}(p-r)+\frac{c}{r}(p-q)=0$,
or, $\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$.

Ex. 13. The sum of n terms of two Arithmetic series are in the ratio of $n-1:n+1$; find the ratio of their 5th terms.

Let the two series respectively have a and f as their first terms and b and d as their common differences.

We have to find the ratio of their 5th terms *i.e.*, $\frac{a+4b}{f+4d}$.

From the given data we have $\dfrac{\frac{n}{2}\{2a+(n-1)b\}}{\frac{n}{2}\{2f+(n-1)d\}}=\dfrac{n-1}{n+1}$, or, $\dfrac{2a+(n-1)b}{2f+(n-I)d}=\dfrac{n-1}{n+1}$. Now, putting 9 for n we have

$\dfrac{2a+8b}{2f+8d}=\dfrac{9-1}{9+1}$, or, $\dfrac{2(a+4b)}{2(f+4d)}=\dfrac{8}{10}$, $\therefore$ $\dfrac{a+4b}{f+4d}=\dfrac{4}{5}$.

$\therefore$ The reqd ratio $=4:5$.

Ex. 14. If s_1, s_2, s_3 denote the sums of n terms of three series in A. P., the first term of each being the same and the respective common differences 1, 2 and 3, show that $s_1+s_3=2s_2$.

Let a be the first term.

$$\begin{aligned}\therefore\quad s_1&=\tfrac{n}{2}\{2a+(n-1)\times 1\}=\tfrac{n}{2}(2a+n-1),\\ s_2&=\tfrac{n}{2}\{2a+(n-1)\times 2\}=\tfrac{n}{2}(2a+2n-2),\\ s_3&=\tfrac{n}{2}\{2a+(n-1)\times 3)=\tfrac{n}{2}(2a+3n-3).\\ \therefore\quad s_1+s_3&=\tfrac{n}{2}(2a+n-1)+\tfrac{n}{2}(2a+3n-3)\\ &=\tfrac{n}{2}(4a+4n-4)=2\times\tfrac{n}{2}(2a+2n-2)=2s_2.\end{aligned}$$

Ex. 15. The angles of a rectilineal figure are in A. P. If the least of them be 88° and their common difference 10°, find the number of sides.

Let n be the number of sides. Here $a=88^\circ$, $b=10^\circ$.

$\because$ The interior angles of a rectilineal figure $+4$ rt. angles $=$ twice as many rt. angles as the figure has sides,

$\therefore$ $\frac{n}{2}\{2.88^\circ+(n-1).10^\circ\}+360^\circ=2.n.90^\circ$,

or, $88n+5n^2-5n+360=180n$,

or, $5n^2-97n+360=0$, or, $5n^2-72n-25n+360=0$,

or, $(n-5)(5n-72)=0$, $\therefore$ $n=5$ or $\frac{72}{5}$.

$\because$ the number of sides cannot be a fraction,

$\therefore$ the reqd. number of sides $=5$.

Exercise 4.

Find the sum of :—

1. $1+3+5+7+\ldots$to 10 terms. [C. U. '23]

2. $5+8+11+\ldots$to n terms.

3. $14+10+6+\ldots$to 12 terms.

4. $-14-11-8\ldots$to 20 terms.

5. $1+1\frac{1}{4}+1\frac{1}{2}+$to 19 terms.

6. $1+\frac{n-1}{n}+\frac{n-2}{n}+\ldots$to n terms.

7. $12+12{\cdot}6+13{\cdot}2+\ldots$to 21 terms.

8. $1+5+9+\ldots$to $(n-1)$ terms.

9. $n+(n-1)+(n-2)+\ldots$to $(n+2)$ terms.

10. $\sqrt{2}+\sqrt{2}(1+\sqrt{2})+\sqrt{2}(1+2\sqrt{2})+\ldots$to 19 terms.

11. $2+5+8+\ldots+152$. [C. U. '48]

12. $9+7+5+(-25)$.

Find, without assuming any formula, the sum of

13. 1, 3, 5, 7,...to 30 terms. [C. U. '16]

14. $4+7+10+\ldots$to 112 terms. [D. B. '44]

15. $5+8+11+\ldots$to 51 terms. [E. B. S. B. '51[

16. $3+7+11+\ldots$to n terms.

17. $1+4+7+10+\ldots+37$. [C. U. '19]

18. Find, without assuming any formula, the sum of the first n natural numbers. [C. U. '10, '19 ; D. B. '28]

19. Find the sum of all the multiples of 7 between 320 and 442.

20. Find the sum of all the odd numbers between 100 and 200.

21. Find the sum of 30 consecutive odd numbers of which the last is 127.

22. Find the sum of 21 terms of an A. P. of which the 4th and 15th terms are 13 and 57 respectively.

23. Find the sum of 16 terms of the series 10, 8, 6,..., beginning at the 10th term.

24. The 21st term of an A. P. is 43, find the sum of its first 41 terms.

25. How many terms must be taken of the series 2, 8, 14,... to make the sum 352 ? [C. U. '49]

26. The first two terms of an A. P. are $1\frac{1}{2}$ and $2\frac{1}{3}$. How many terms of the series must be taken to give the sum 171 ?

[D. B. '40]

27. Find the sum of $1+4+6+9+11+14+\ldots$to 21 terms.

28. The sum of n terms of an A. P. is n^2. Find the first term and the common difference. [G. U. '48]

29. Find the series in A. P. of which the sum of r terms is $2r^2+3r$.

30. The sum of a certain number of terms of the A. P. $21+19+17+\ldots$ is 120. Find the last term and the number of terms. [D. B. '47]

31. The sum of n terms of a series is $\dfrac{n(4n^2-1)}{3}$. Find the rth term.

32. The sum of 9 terms of an A. P. is 171 and that of 24 terms is 996. Find the sum of 41 terms.

33. The sum of n terms of an A. P. is m, and that of m terms is n. Prove that the sum of $m+n$ terms is $-(m+n)$. [C. U. '50]

34. Show that if unity be added to the sum of any number of terms of the series 8, 16, 24,..., the result will be the square of an odd number.

35. The sum of 3 numbers in A. P. is 36 and their product is 1140 : find the numbers.

36. Find four numbers in A. P. of which the sum is 22 and the product of the extreme terms is 10.

37. Find the sum of n Arithmetic means between a and c.

38. Prove that the sum of n Arithmetic means between two quantities is n times the single mean between them.

39. Sum to n terms the series whose nth term is $2n-1$.

[D. B. '46]

Sum to n terms :

40. $1\times3+3\times5+5\times7+7\times9+\ldots$ [W. B. S. F. '52]

41. $3.7+5.10+7.13+9.16+\ldots$ [D. B. '36]

42. $5^2+8^2+11^2+\ldots$ **43.** $1.2^2+2.3^2+3.4^2+\ldots$

44. $2.3+3.4+4.5+\ldots$ [E. B. S. B. '49]

45. $1+3+6+10+15+\ldots$ **46.** $1+4+9+16+25+\ldots\ldots$

47. $2.3.4+3.4.5+4.5.6+\ldots$ **48.** $(1)+(1+3)+(1+3+5)+\ldots$

49. $\frac{1}{2.4}+\frac{1}{4.6}+\frac{1}{6.8}+\ldots$ **50.** $\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}+\ldots$

51. (*a*) $1.7+3.9+5.11+\ldots$ [C. U. High '50]

51. (*b*) $1^2-2^2+3^2-4^2+5^2-6^2+\ldots$to r terms.

52. If x, y, z are in A. P., shew that $y+z$, $z+x$, $x+y$ are in A. P.

53. If $\frac{b+c}{a}$, $\frac{c+a}{b}$, $\frac{a+b}{c}$ are in A. P., prove tnat $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are in A. P.

54. If $(b-c)^2$, $(c-a)^2$, $(a-b)^2$ are in A. P., show that $\frac{1}{b-c}$, $\frac{1}{c-a}$, $\frac{1}{a-b}$ are in A. P.

55. Four numbers are in A. P. The sum of their extremes is 11, while the product of the means is $29\frac{1}{4}$. Find the numbers. [D. B. '35]

56. If a, b, c are in A. P., show that
$(a+2b-c)(2b+c-a)(c+a-b)=4abc.$ [D. B. '31]
[Hints :—$\because$ a, b, c are in A. P., $\therefore$ $a+c=2b$]

57. If x, y, z be respectively the sums of the first p, q and r terms of a series in A. P., prove that $xqr(q-r)+yrp(r-p)+zpq(p-q)=0.$ [C. U.]

58. A person lends Rs. 1000 to a friend agreeing to charge no interest and also to recover the amount by monthly instalments decreasing successively by Rs. 2. In how many months will the loan be paid up, if the first instalment be Rs. 64 ? [C. U. '20]

59. The vertical angles of a polygon are in A.P. The smallest angle is 120° and the common difference is 5°. Find the number of sides of the polygon.

60. 100 stones are placed on a straight road at intervals of 5 yds. apart. A runner has to start from a basket 5 yds. from the first stone, pick up the stones and bring them back to the basket one by one. How many yards has he to run altogether ? [Pat. U. '19]

61. A man has to travel 162 miles : he goes 30 miles the first day, 27 the second, 24 the third, and so on. How many days does he take for the journey ? [D. B. '24]

62. A tree in each year grows 1 inch less than it did in the previous year. If it grew 1 yd. in the first year, in how many years will it have ceased growing and what is its height then ?

63. A man undertakes to pay off a debt of Rs. 65 by monthly instalments ; he pays Rs. 2 in the first month and continually increases the instalments in subsequent month by Re. 1. In what time will the debt be cleared up ? [C. U. '30, '50]

64. A sets out from a place and travels at the rate of 5 miles an hour. B sets out $4\frac{1}{2}$ hours ofter A and travels in the same direction, 3 miles the first hour, $3\frac{1}{2}$ miles the second hour, 4 miles the third hour and so on. Find in how many hours B will overtake A. [H. S. '65]

GEOMETRICAL PROGRESSION

9. **Geometric Series.** A series is said to be in Geometrical Progression, if the ratio of any term to the preceding one is the same throughout the series.

Hence in a geometric series each term is formed by multiplying the preceding one by a constant factor.

The ratio (or the constant factor) is called the **common ratio.** It is found by dividing any term by the term preceding it.

Geometrical Progression is briefly written as G. P.

Thus, (i) 1,3,9,27,...are in G. P. and the common ratio$=\frac{3}{1}=3$.

(ii) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}$,...are in G. P. and the com. ratio

$$=-\tfrac{1}{2}\div 1=-\tfrac{1}{2}.$$

[**N. B.** (i) The common ratio is generally found by dividing the second term by the first term and it is denoted by r.

(2) In a G. P. the successive terms from the beginning are formed by multiplying the preceding term by the common ratio. (3) So from the end of the series the preceding terms are formed by dividing each term by the common ratio. (4) The terms of a series in G. P. are evidently in continued proportion.]

10. **General term.** If a be the first term and r the common ratio of a series in G.P., then evidently the series is a, ar, ar^2, ar^3,.... This is the standard form of a G.P.

Here the first term $t_1=a$, i.e., ar° i.e. $a.r^{1-1}$,

the second term $t_2=ar$, i.e., $a.r^{2-1}$,

the third term $t_3=ar^2$, i.e., $a.r^{3-1}$,

and so on.

Here we notice that in each term the power of r is one less than the serial number of the term.

$\therefore$ evidently the nth term or $\mathbf{t_n=ar^{n-1}}$, it is the general term of the series.

Examples (7)

Ex. 1. Find the 6th term of the series 1, 2, 4, 8,...

Here, the 1st term $a=1$, the common ratio $r=\frac{2}{1}=2$,

and the number of terms $n=6$.

$\because \quad t_n=ar^{n-1}, \ \therefore \ t_6=1.2^5=32.$

Ex. 2. Find the 8th term of the series $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots$

Here $a=1$, $r=\dfrac{-\frac{1}{3}}{1}=-\frac{1}{3}$ and $n=8$.

$$\because \quad t_n=ar^{n-1}, \ \therefore \ t_8=1\times(-\tfrac{1}{3})^7=-\frac{1}{3^7}=-\frac{1}{2187}.$$

Ex. 3. Find the *pth* term of the series 4, 8, 16,...

Here, $a=4$, $r=\frac{8}{4}=2$ and $n=p$.

$\because \quad t_n=ar^{n-1}, \ \therefore \ t_p=4.2^{p-1}=2^2.2^{p-1}=2^{p+1}.$

Ex. 4. Find the *n*th term of the series $1, -3, 9, -27, \ldots$

Here $a=1$, $r=\frac{-3}{1}=-3$.

$\therefore \quad t_n=ar^{n-1}=1.(-3)^{n-1}=(-3)^{n-1}.$

[**N.B.** Here it is not known whether t_n, i.e., $(-3)^{n-1}$ is positive or negative. If n be an even integer, then $n-1$ is odd and consequently $(-3)^{n-1}$ is negative. If n be odd, $n-1$ will be even and hence $(-3)^{n-1}$ will be positive.]

Ex. 5. Find the *n*th term of the series

$$\sqrt{3}+\frac{1}{\sqrt{3}}+\frac{1}{3\sqrt{3}}+\ldots\ldots \qquad \text{[C. U.]}$$

$$\text{Here } a=\sqrt{3}, \text{ and } r=\frac{\frac{1}{\sqrt{3}}}{\sqrt{3}}=\frac{1}{\sqrt{3}\times\sqrt{3}}=\frac{1}{3}.$$

$$\therefore \quad t_n=ar^{n-1}=\sqrt{3}.\,(\tfrac{1}{3})^{n-1}=\frac{\sqrt{3}}{3^{n-1}}=3^{\frac{1}{2}}.3^{1-n}=3^{\frac{3}{2}-n}.$$

Ex. 6. Find the common ratio of the G. P. of which the first term is 2 and the 10th term is 1. [C. U. '25]

Here $a=2$, $t_{10}=1$; Let r be the *common ratio.*

$\because \quad t_n=ar^{n-1}, \ \therefore \ t_{10}=2r^9, \ \text{or}, \ 1=2r^9$

or, $r^9=\frac{1}{2}. \ \therefore \ r=\sqrt[9]{\tfrac{1}{2}}.$

Ex. 7. Find the 9*th* term of the G. P. of which the 4*th* and 11*th* terms are 2 and $\frac{1}{64}$ respectively.

Let a be the first term and r the common ratio.

Then from the given conditions $t_4 = 2$, *i.e.*, $ar^3 = 2 \ldots (1)$

and $t_{11} = \frac{1}{64}$, *i.e.*, $ar^{10} = \frac{1}{64} \ldots (2)$.

Now, from $(2) \div (1)$ we get $r^7 = \frac{1}{128} = (\frac{1}{2})^7$, $\therefore \quad r = \frac{1}{2}$.

$\therefore$ From (1) we get $a(\frac{1}{2})^3 = 2$, or, $\frac{1}{8}a = 2$, $\therefore \quad a = 16$.

$$\therefore \quad \text{the reqd. } t_9 = ar^8 = 16 \times \frac{1}{2^8} = 2^4 \times \frac{1}{2^8} = \frac{1}{2^4} = \frac{1}{16}.$$

Ex. 8. Find the series in G. P. whose 5th term is 16 and 9th term is 256.

Let a be the first term and r the common ratio.

Then from the given data we have,

$$ar^4 = 16 \ldots (1) \text{ and } ar^8 = 256 \ldots (2).$$

Dividing (2) by (1) we have $r^4 = 16$, $\therefore \quad r = \pm 2$.

Now, from (1) we have $a.16 = 16$, $\therefore \quad a = 1$.

$\therefore$ the required series is 1, 2, 4, 8,... ; or, 1, -2, 4, -8,...

Ex. 9. Which term of the series 9, 3, 1,...is $\frac{1}{243}$?

Let $\frac{1}{243}$ be the nth term of the series.

Here $a = 9$ and $r = \frac{9}{3} = \frac{1}{3}$.

$\therefore \quad ar^{n-1} = \frac{1}{243}$, or, $9.(\frac{1}{3})^{n-1} = \frac{1}{243}$.

$$\text{or,} \quad 3^2 \times \frac{1}{3^{n-1}} = \frac{1}{243}, \quad \text{or,} \quad \frac{1}{3^{n-3}} = \frac{1}{3^5}, \quad \text{or,} \quad 3^{n-3} = 3^5,$$

$\therefore \quad n - 3 = 5$, $\therefore$ $n = 8$. $\therefore$ $\frac{1}{243}$ is the 8th term of the series.

Ex. 10. If there are 6 terms in a G. P., prove that the product of the first and last is equal to the product of the third and fourth. [P. U.]

Let a be the 1st term and r the *common ratio*.

Here the last term = the 6th term $= ar^5$.

$\therefore$ the first term × the last term $= a \times ar^5 = a^2r^5$.

Again, the third term × the fourth term $= ar^2 \times ar^3 = a^2r^5$.

Hence the product of the 1st and last terms = the product of the 3rd and 4th terms.

Ex. 11. Show that the product of any two terms of a G. P., equidistant from the beginning and the end, is constant.

[D. B. '31]

Let a be the first term, b the last term and r the common ratio.

Let us take the pth terms from the beginning and the end as two equidistant terms.

The pth term from the beginning $=ar^{p-1}$,

and the (preceding) pth term from the end $=\frac{b}{r^{p-1}}$.

$\therefore$ the product of the two equidistant pth terms

$=ar^{p-1}\times\frac{b}{r^{p-1}}=ab$ which is constant.

Ex. 12. If the pth and the qth terms of a G. P. be c and d respectively, find the first term and the common ratio.

[C. U. '35]

Let a be the 1st term and r the common ratio.

Then from the given conditions we have

$$ar^{p-1}=c\ldots(1) \text{ and } ar^{q-1}=d\ldots(2)$$

Now, from (1)÷(2) we get $\frac{ar^{p-1}}{ar^{q-1}}=\frac{c}{d}$ or, $r^{p-q}=\frac{c}{d}$, $\therefore\ r=\left(\frac{c}{d}\right)^{\frac{1}{p-q}}$

$\therefore$ from (1) we get $a\left(\frac{c}{d}\right)^{\frac{p-1}{p-q}}=c$, $\therefore\ a=c\times\left(\frac{d}{c}\right)^{\frac{p-1}{p-q}}=c\times\frac{d^{\frac{p-1}{p-q}}}{c^{\frac{p-1}{p-q}}}$

$=c^{1-\frac{p-1}{p-q}}\times d^{\frac{p-1}{p-q}}=c^{\frac{1-q}{p-q}}.d^{\frac{p-1}{p-q}}=(c^{1-q}.d^{p-1})^{\frac{1}{p-q}}$.

Ex. 13. In a G. P. if the $(p+q)$th term is m and $(p-q)$th term is n, find the pth and qth terms.

[B. U. 1888 ; C. U. '34, '42]

Let a be the 1st term and r be the common ratio.

$\therefore\ t_{p+q}=m$, *i.e.*, $ar^{p+q-1}=m\ldots(1)$

and $t_{p-q}=n$, *i.e.*, $ar^{p-q-1}=n\ldots(2)$. Now, dividing (1) by (2) we have $r^{p+q-1-p+q+1}=\frac{m}{n}$, or, $r^{2q}=\frac{m}{n}$, $\therefore\ r=\left(\frac{m}{n}\right)^{\frac{1}{2q}}$

Now, from (1), $a\left\{\left(\frac{m}{n}\right)^{\frac{1}{2q}}\right\}^{p+q-1}=m$, or, $a\left(\frac{m}{n}\right)^{\frac{p+q-1}{2q}}=m$

$$\therefore \quad a=m.\left(\frac{n}{m}\right)^{\frac{p+q-1}{2q}}.$$

$\therefore$ the reqd. $t_p=ar^{p-1}=m.\left(\frac{n}{m}\right)^{\frac{p+q-1}{2q}}\times\left\{\left(\frac{m}{n}\right)^{\frac{1}{2q}}\right\}^{p-1}$

$$=m.\left(\frac{n}{m}\right)^{\frac{p+q-1}{2q}}.\left(\frac{m}{n}\right)^{\frac{p-1}{2q}}=m.\left(\frac{n}{m}\right)^{\frac{p+q-1}{2q}}.\left(\frac{n}{m}\right)^{\frac{1-p}{2q}}$$

$$=m.\left(\frac{n}{m}\right)^{\frac{p+q-1+1-p}{2q}}=m.\left(\frac{n}{m}\right)^{\frac{q}{2q}}=m.\left(\frac{n}{m}\right)^{\frac{1}{2}}=m\times\frac{n^{\frac{1}{2}}}{m^{\frac{1}{2}}}$$

$=m^{\frac{1}{2}}.n^{\frac{1}{2}}=\sqrt{mn}$. Similarly $t_q=ar^{q-1}=m.\left(\frac{n}{m}\right)^{\frac{p}{2q}}$.

[**N. B.** $\left(\frac{m}{n}\right)^{\frac{p-1}{2q}}=\left(\frac{m}{n}\right)^{-\frac{1-p}{2q}}=\frac{1}{\left(\frac{m}{n}\right)^{\frac{1-p}{2q}}}=\left(\frac{n}{m}\right)^{\frac{1-p}{2q}}$]

Exercise 5

1. Find the 6th term of the series 4, 8, 16,...

2. Find the 9th term of 1, -3, 9,...

3. Find the 7th term of $\frac{5}{4}$, 1, $\frac{4}{5}$,...

4. Find the nth term of 1, $\frac{1}{3}$, $\frac{1}{9}$,...

5. What is the nth term of 16, 8, 4,... ?

6. Find the pth term of the series 3, -9, 27,...

7. Find the 7th term of the G. P. of which the first term is 9 and the common ratio is $-\frac{1}{3}$.

8. The common ratio and the 6th term of a G.P. are $\frac{1}{2}$ and 1 respectively. Find its 11th term.

9. The 4th term of a G. P. is 8 and the 9th term is 256, find the series.

10. The 10th term of a G. P. is $\frac{1}{2}$ and the 14th term is $\frac{1}{32}$, find the 17th term.

11. The first two terms of a G. P. are 3 and 1. Write down the 10th term. [C. U. '13]

12. The 5th term of a G. P. is 48 and the 12th term 6144. Find the 1st term and the common ratio. [D. B. '28]

13. Which term of the series 128, 64, 32,...is $\frac{1}{2}$?

14. Is 320 a term of the series 5, -10, 20,... ?

15. Find the 10th and nth terms of a G. P. of which the rth term is 2^{r-1}.

16. If the pth and qth terms of a G. P. are a and b respectively, find its nth term.

Geometric Mean

11. If three quantities are in G. P., the middle one is called the **Geometric Mean** of the other two.

It is generally denoted by G. M. If any number of quantities are in G. P., all the terms between the first and the last are called the geometric means between the first and the last terms.

Thus, 3, 9, 27 being in G. P., 9 is the G.M. between 3 and 27. If $a, ar, ar^2, ar^3, \ldots\ldots, ar^n, b$ are in G.P., then $ar, ar^2, ar^3, \ldots\ldots, ar^n$ are the n geometric means between a and b.

Examples (8)

Ex. 1. Find the geometric mean between a and b. [C. U. '48]

Let m be the G. M. between a and b ; then a, m, b are in G.P.

$$\therefore \frac{m}{a}=\frac{b}{m} \quad [\because \text{ each}=\text{common ratio}]$$

$$\text{or, } m^2=ab, \quad \therefore \ m=\pm\sqrt{ab}.$$

[**Note** that the G. M. between two given quantities is the square root of their product.]

Ex. 2. Insert n geometric means between a and b.

Here the n means together with a and b at the two ends will form a G. P. of $(n+2)$ terms, of which a is the 1st term and b is the $(n+2)$th term.

Let the common ratio be r.

$$\therefore \quad b = ar^{n+1}, \quad \text{or,} \quad r^{n+1} = \frac{b}{a}, \quad \therefore \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}.$$

$$\therefore \quad \text{The means are } a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, \ldots\ldots, a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}.$$

Ex. 3. Insert 3 geometric means between 2 and 162.

[C. U. '30, '49]

If 3 G. M.'s are inserted between 2 and 162, we get 5 terms in G. P. of which the first term is 2 and the fifth term is 162.

Let r be the common ratio.

$\because \quad t_5 = 162, \quad \therefore \quad ar^4 = 162, \quad \text{or,} \quad 2r^4 = 162,$

or, $r^4 = 81 = (\pm 3)^4, \quad \therefore \quad r = \pm 3.$

$\therefore$ The 1st mean $= 2 \times 3$ or $2 \times -3 = 6$ or -6

The 2nd mean $= 6 \times 3$ or $-6 \times -3 = 18$ or 18

The 3rd mean $= 18 \times 3$ or $18 \times -3 = 54$ or -54

$\therefore$ The 3 means are 6, 18, 54 or -6, 18, -54.

Ex. 4. Insert 5 geometric means between $3\frac{5}{9}$ and $40\frac{1}{2}$.

[D.B. '35]

Here the 5 means together with $3\frac{5}{9}$ and $40\frac{1}{2}$ at the two ends will form a G. P. of which the first term is $3\frac{5}{9}$ and the number of terms $= 5 + 2 = 7$.

Let r be the common ratio.

Now, $t_7 = 40\frac{1}{2}$, $a = 3\frac{5}{9} = \frac{32}{9}$, $\quad \therefore \quad ar^{7-1} = 40\frac{1}{2}$,

$$\text{or,} \quad \frac{32}{9}r^6 = \frac{81}{2}, \quad \text{or,} \quad r^6 = \frac{81 \times 9}{2 \times 32} = \frac{3^6}{2^6} = \left(\frac{3}{2}\right)^6 \quad \therefore \quad r = \pm\tfrac{3}{2}.$$

$\therefore$ The reqd. means $= 5\frac{1}{3}$, 8, 12, 18, 27 ;

or, $-5\frac{1}{3}$, 8, -12, 18, -27.

Ex. 5. The arithmetic mean between two numbers is 15 and their geometric mean is 9. Find the numbers. [C. U. '26]

Let a and b be the numbers.

$\because$ the arithmetic mean between a and $b=\frac{a+b}{2}$,

$\therefore$ here $\frac{a+b}{2}=15$, or, $a+b=30$...(1).

Again, $\because$ the G. M. between a and $b=\pm\sqrt{ab}$,

$\therefore$ here $\pm\sqrt{ab}=9$, $ab=81$...(2).

Now, $(a-b)^2=(a+b)^2-4ab=30^2-4\times81=900-324=576$,

$\therefore$ $a-b=\pm\sqrt{576}=\pm24$...(3).

Solving (1) and (3) we have $a=27$ and $b=3$,

or, $a=3, b=27$ [if $a-b=-24$]

$\therefore$ The reqd. numbers are 27 and 3, or 3 and 27.

Ex. 6. Show that the arithmetic mean of any two real positive quantities is greater than their geometric mean.

[C. U. '28, '39, '41, '44, '47, '48 : G. U. '52]

Let a and b be two real positive quantities.

The A.M. between a and $b=\frac{a+b}{2}$, and their G.M. $=\pm\sqrt{ab}$.

Now, we have $\frac{a+b}{2}-(\pm\sqrt{ab})=\frac{a+b\mp2\sqrt{ab}}{2}=\frac{(\sqrt{a}\mp\sqrt{b})^2}{2}$,

here the numerator being a perfect square is positive, as a and b are real. $\therefore$ $\frac{a+b}{2}>\pm\sqrt{ab}$.

$\therefore$ the arithmetic mean is greater than the geometric mean.

Ex. 7. Prove that the product of the n geometric means between a and b is the nth power of the single mean between them.

Let r be the common ratio.

The G. M. between a and $b=\sqrt{ab}=(ab)^{\frac{1}{2}}$

$\therefore$ The nth power of the mean $=\{(ab)^{\frac{1}{2}}\}^n=(ab)^{\frac{n}{2}}$.

The n G.M.'s between a and $b=ar, ar^2, ar^3, \ldots\ldots, \frac{b}{r^3}, \frac{b}{r^2}, \frac{b}{r}$.

Let p be their product. We are to prove that $p=(ab)^{\frac{n}{2}}$.

Now, $p=ar\times ar^2\times ar^3\times\ldots\times\frac{b}{r^3}\times\frac{b}{r^2}\times\frac{b}{r}$

Also $p=\frac{b}{r}\times\frac{b}{r^2}\times\frac{b}{r^3}\times\ldots\times ar^3\times ar^2\times ar$

(writing in reverse order)

$\therefore$ (multiplying), $p^2=ab\times ab\times ab\times\ldots$to n factors$=(ab)^n$,

$\therefore\quad p=(ab)^{\frac{n}{2}}$.

[**N. B.** Here $\because$ a is the 1st term, b the last term and r the com. ratio, $\therefore$ from the beginning ar is the 1st mean, ar^2 the second mean and so on. Again, from the end the last mean is the term preceding the last term b, so the last mean may be written as $\frac{b}{r}$, the mean preceding $\frac{b}{r}$ is $\frac{b}{r^2}$, and so on. In multiplication above $\left(ar\times\frac{b}{r}\right)$, $\left(ar^2\times\frac{b}{r^2}\right)$, $\left(ar^3\times\frac{b}{r^3}\right)$ and so on have been multiplied together]

Ex. 8. A is the arithmetic mean and G the geometric mean of two unequal positive real numbers p and q. Prove that $A>G>\frac{G^2}{A}$. [G. U. '50]

Here $A=$the A. M. between p and $q=\frac{p+q}{2}$

and $G=$the G. M. of p and $q=\sqrt{pq}$

$\therefore\quad \frac{G^2}{A}=\frac{pq}{\frac{p+q}{2}}=\frac{2pq}{p+q}=k$ (suppose).

Now $\frac{p+q}{2}-\sqrt{pq}=\frac{p+q-2\sqrt{pq}}{2}=\frac{1}{2}(\sqrt{p}-\sqrt{q})^2$;

as here p and q are real, $\frac{1}{2}(\sqrt{p}-\sqrt{q})^2$ is positive,

$\therefore\quad \frac{p+q}{2}>\sqrt{pq}$, *i.e.*, $A>G$ (proved).

Again, $A.k = \frac{p+q}{2} \times \frac{2pq}{p+q} = pq = G^2 = G.G.$

but $A > G$ (already proved),

$\therefore\ k < G$, *i.e.*, $G > \frac{G^2}{A}$. $\qquad \therefore\ A > G > \frac{G^2}{A}$.

Ex. 9. The A. M. of a and b is to their G. M. as m is to n. Shew that $a : b = m + \sqrt{m^2 - n^2} : m - \sqrt{m^2 - n^2}$. [A.U. 1889]

A. M. between a and $b = \frac{a+b}{2}$ and their G. M. $= \sqrt{ab}$

$\therefore\ \frac{a+b}{2} : \sqrt{ab} = m : n$ (given),

or, $\frac{a+b}{2\sqrt{ab}} = \frac{m}{n}$, or, $\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{m+n}{m-n}$ (by comp. & div.)

or, $\frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2} = \frac{m+n}{m-n}$, or, $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{m+n}}{\sqrt{m-n}}$

(taking sq. root)

or, $\frac{2\sqrt{a}}{2\sqrt{b}} = \frac{\sqrt{m+n}+\sqrt{m-n}}{\sqrt{m+n}-\sqrt{m-a}}$ (by com. & div.)

or, $\frac{a}{b} = \frac{m+n+m-n+2\sqrt{m^2-n^2}}{m+n+m-n-2\sqrt{m^2-n^2}}$ (squaring)

or, $\frac{a}{b} = \frac{2(m+\sqrt{m^2-n^2})}{2(m-\sqrt{m^2-n^2})}$,

$\therefore\ a : b = m + \sqrt{m^2-n^2} : m - \sqrt{m^2-n^2}$.

Ex. 10. Show that if p and q are two unequal positive numbers then $A > G > H$, where A is the Arithmetic mean and G is the Geometric mean and $H = \frac{2}{\frac{1}{p}+\frac{1}{q}}$. [E. B. S. B. '48]

Here, $A = \frac{p+q}{2}$, $G = \sqrt{pq}$ and $H = \frac{2}{\frac{1}{p}+\frac{1}{q}} = \frac{2}{\frac{p+q}{pq}} = \frac{2pq}{p+q}$.

[Now proceed as in Ex. 8 above]

Exercise 6

Find the geometric mean between :—

1. 5 and 125. **2.** $\frac{1}{2}$ and $\frac{1}{8}$. **3.** -3 and -27.

4. -6 and $-2\frac{2}{3}$. **5.** x^3y and xy^3 ; $3\sqrt{3}$ and $9\sqrt{3}$.

6. Insert 3 geometric means between 4 and 324. [C. U. 1890]

7. Insert 2 geometric means between 5 and 135. [C. U. '16]

8. Insert 3 geometric means between 25 and 164025. [Pat. U. '19]

9. Insert 3 geometric means between $\frac{1}{9}$ and 9. [C. U. '14]

10. Insert 9 geometric means between $\frac{81}{16}$ and $\frac{64}{729}$. [D. B. '30]

11. Show that the $2n$th term of any G. P. is the mean proportional between the nth and $3n$th terms. [C. U. 1877]

12. In a G. P., show that the product of any two terms equidistant from a given term is equal to the square of the given term. [C. U. '15]

[**Hints** : Let a be the 1st term, r the common ratio and let the given term be the mth term. Then $t_m = ar^{m-1}$ and its square $= (ar^{m-1})^2$. Now let us take the pth term preceding and the pth term following the mth term. Evidently the pth term preceding the mth term $= \dfrac{ar^{m-1}}{r^p} = ar^{m-1-p}$, and the pth term following the mth term $= ar^{m-1} \times r^p = ar^{m-1+p}$.

$\therefore$ The product of the two pth terms $= ar^{m-1-p} \times ar^{m-1+p}$

$= a^2r^{2m-2} = (ar^{m-1})^2$.]

13. If the A. M. and G. M. of two quantities be respectively A and G, show that the quantities are $A \pm \sqrt{A^2 - G^2}$.

14. If there are an odd number of terms in a G. P., show that the product of the first and the last terms is equal to the square of the middle term.

15. If the number of terms in a G. P. be even, prove that the product of its two middle terms is equal to the product of its first and last terms.

[Let the number of terms be $2n$. $\therefore$ the nth and the $(n+1)$th terms will be the two middle terms.]

16. Find two numbers such that their A. M. is 25 and G. M. 24.

12. Sum of a Series in G. P.

Examples (9)

Ex. 1 Find the sum of the first n terms of a G. P.
[C. U. '19 '29. '39, '40, '42 : D. B. '32]

Or, Find the sum of n terms of a G. P., being given the first term and the common ratio. [C. U. '17]

Or, Find the sum of the first n terms of a G. P., the first term being a and the common ratio r.
[C. U. '32, '35, '37, '46 ; D. B. '39, '42]

Or, Find the sum of $a+ar+ar^2+ar^3+$......to n terms.
[C. U. '31 ; D, B. '36]

Let a be the first term, r the common ratio and s the sum. Here the number of terms $=n$, so the nth term $=ar^{n-1}$.

$$\therefore \quad s=a+ar+ar^2+...+ar^{n-1} \text{............} (1)$$

$$\underline{\text{and from } (1)\times r, \; sr= \quad ar+ar^2+...+ar^{n-1}+ar^n \text{......}(2)}$$

$\therefore$ Subtracting we get $s-sr=a-ar^n$,

$$\text{or,} \quad s(1-r)=a(1-r^n), \quad \therefore \quad \mathbf{s=\frac{a(1-r^n)}{1-r}} \text{...}(3)$$

Again from (2) - (1) we have $sr - s = ar^n - a$.

$$\therefore \quad \mathbf{s} = \frac{\mathbf{a(r^n - 1)}}{\mathbf{r - 1}} \ldots (4)$$

[**N. B.** (*i*) Of the two formulas (3) and (4), we apply (4) when r is positive and greater than 1, and we apply (3) in other cases, *i. e.* when $r<1$ or negative. (*ii*) If l be the last term, then $l = ar^{n-1}$,

$\therefore$ from (4) we have $s = \dfrac{ar^n - a}{r-1} = \dfrac{ar^{n-1}.r - a}{r-1} = \dfrac{lr - a}{r-1}$, this formula is applied when the last term is known.]

Ex. 2. Find the sum of $1+2+4+\ldots$to 20 terms. [C. U. '22]

Here the 1st term $a=1$, the common ratio $r=\frac{2}{1}=2$ and the number of terms $n=20$. Let s be the sum.

$$\therefore \quad s = \frac{a(r^n - 1)}{r-1} = \frac{1(2^{20} - 1)}{2-1} = 2^{20} - 1.$$

Ex. 3. Sum the series $1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots$to n terms.

[C. U. '12 ; '39 Sup.]

Here, $a=1$, $r=\frac{1}{3} \div 1 = \frac{1}{3}$. Let s be the sum.

$$\therefore \quad s = \frac{a(1-r^n)}{1-r} = \frac{1\{1-(\frac{1}{3})^n\}}{1-\frac{1}{3}} = \frac{1 - \dfrac{1}{3^n}}{\frac{2}{3}} = \frac{3}{2}\left(1 - \frac{1}{3^n}\right).$$

Ex. 4. Find the sum of $1 - 3 + 9 - 27 + \ldots$to 13 terms.

Here, $a=1$, $r = -3 \div 1 = -3$, $n=13$, let s be the sum.

$$\therefore \quad s = \frac{a(1-r^n)}{1-r} = \frac{1\{1-(-3)^{13}\}}{1-(-3)} = \frac{1+3^{13}}{4} = \tfrac{1}{4}(1+3^{13}).$$

Ex. 5. Find the sum of $1 - 2 + 4 - 8 + \ldots$to n terms.

Here, $a=1$, and $r=-2$. Let s be the sum.

$$\therefore \quad s = \frac{a(1-r^n)}{1-r} = \frac{1\{1-(-2)^n\}}{1-(-2)} = \frac{\{1-(-2)^n\}}{3} = \tfrac{1}{3}\{1-(-2)^n\}.$$

[**N. B.** In Ex. 4 above, $(-3)^{13}$ is negative as its power (13) is an odd number. But in Ex. 5, it cannot be definitely known whether $(-2)^n$ is positive or negative. It will be positive if n be even and negative if n be odd.]

Ex. 6. Find the sum of the series $3+6+12+\ldots+384$.

Here $a=3$, $r=6\div3=2$, and the last term $l=384$.

$$\therefore \text{ the reqd. sum}=\frac{lr-a}{r-1}=\frac{2\times384-3}{2-1}=765.$$

Ex. 7. Find, without assuming any formula, the sum of $1+4+16+\ldots$to 10 terms.

Here $a=1$, $r=4\div1=4$.

Now, $t_1=1$, $t_2=4=1.4^1$, $t_3=16=1.4^2$.

$\therefore$ evidently $t_{10}=1.4^9=4^9$. Let s be the sum.

$\therefore\ s=1+4+4^2+\ldots\ldots+4^9\ldots\ldots(1)$

$\therefore\ 4s=\quad 4+4^2+\ldots\ldots+4^9+4^{10}\ldots(2)$

$\therefore$ taking (1) from (2), $3s=4^{10}-1$, $\therefore\ s=\frac{1}{3}(4^{10}-1)$.

Ex. 8. Find, without assuming any formula, the sum of n terms of $1+\frac{1}{2}+\frac{1}{2^2}+\ldots$ [C. U. '10, '18, '23, '38 ; D. B, '34, '40]

Here $t_1=1$, $t_2=\frac{1}{2^1}$, $t_3=\frac{1}{2^2}$ $\therefore$ evidently $t_n=\frac{1}{2^{n-1}}$.

Let s be the sum.

$$\therefore\ s=1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}\ldots\ldots(1)$$

and $\frac{1}{2}s=\quad \frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}+\frac{1}{2^n}$. (2) [Multiplying (1) by $\frac{1}{2}$]

Subtracting (2) from (1), $\frac{1}{2}s=1-\frac{1}{2^n}$, $\therefore\ s=2\left(1-\frac{1}{2^n}\right)$.

Ex. 9. Find the sum of n terms of a G. P. of which the $4th$ term is $\frac{1}{27}$ and the $7th$ term is $\frac{1}{729}$.

Let a be the 1st term and r the common ratio.

$\therefore\ t_4$ or $ar^3=\frac{1}{27}\ldots\ldots(1)$ and t_7 or $ar^6=\frac{1}{729}\ldots(2)$.

From $(2)\div(1)$ we get $ar^6\div ar^3=\frac{1}{729}\div\frac{1}{27}$, or, $r^3=\frac{1}{27}=(\frac{1}{3})^3$,

$\therefore\ r=\frac{1}{3}$. $\therefore$ from (1) we get $a.(\frac{1}{3})^3=\frac{1}{27}$, $\therefore\ a=1$.

$$\text{Now, } s_n=\frac{a(1-r^n)}{1-r}=\frac{1\left(1-\frac{1}{3^n}\right)}{1-\frac{1}{3}}=\frac{3}{2}\left(1-\frac{1}{3^n}\right)=\frac{3^n-1}{2.3^{n-1}}.$$

Ex. 10. Find the sum of 10 terms of the series whose pth term is 3^p+3p.

Here $t_p=3^p+3p$, $\therefore$ putting 1, 2, 3, ..., 10 for p we have

$$t_1=3^1+3.1$$
$$t_2=3^2+3.2$$
$$t_3=3^3+3.3$$
$$\dots\dots\dots\dots$$
$$t_{10}=3^{10}+3.10$$

$\therefore$ the sum $=(3+3^2+3^3+\dots+3^{10})+3(1+2+3+\dots+10)$

$$=\frac{3(3^{10}-1)}{3-1}+3\times\tfrac{10}{2}(10+1)=\tfrac{3}{2}(3^{10}-1)+165=\frac{3^{11}+327}{2}.$$

Ex. 11. Find the sum of $(a-x)+(a^2-x^2)+(a^3-x^3)+\dots+(a^n-x^n)$. [C. U. '30]

We get two series in G. P. from the given series

Let s be the sum

$$\therefore\quad s=(a+a^2+a^3+\dots+a^n)-(x+x^2+x^3+\dots+x^n)$$
$$=(a+a^2+a^3+\dots\text{to } n\text{terms})-(x+x^2+x^3+\dots\text{to } n\text{terms})$$
$$=\frac{a(a^n-1)}{a-1}-\frac{x(x^n-1)}{x-1}.$$

Ex. 12. How many terms of the series 1, 2, 4, ...must be taken so that the sum may be 255 ?

Let 255 be the sum of n terms of the series.

Here $a=1$, $r=2\div1=2$, and $s=255$.

$$\because\quad \frac{a(r^n-1)}{r-1}=s,\quad \therefore\quad \frac{1(2^n-1)}{2-1}=255,\quad \text{or,}\quad 2^n-1=255,$$

or, $2^n=256=2^8$, $\therefore n=8$, $\therefore$ the reqd. number of terms $=8$.

Ex. 13. A certain sum of money produces every year twice as much interest as it did the previous year ; if it produces Rs. 50 in the first year, how much will it produce in 10 yrs. ?

Here the total interest for 10 yrs.

$$=\text{Rs. }(50+100+200+\dots\text{to 10 terms})$$
$$=\text{Rs. }\frac{50(2^{10}-1)}{2-1}=\text{Rs. }50(2^{10}-1)=\text{Rs. }51150.$$

Exercise 7

Find the sum of :—

1. $1+2+4+8+$......to 8 terms. [C. U. '21]

2. $128+64+32+$......to 9 terms.

3. $1+3+9+27+$......to n terms. [C. U. '24, 47]

4. $1-3+9-27+$......to $2n$ terms.

5. $2+1+\frac{1}{2}+$......to 8 terms.

6. $1+\frac{1}{4}+\frac{1}{4^2}+$......to n terms.

7. $\frac{1}{\sqrt{3}}+1+\sqrt{3}+3+$..... to 18 terms.

8. $1+2^{-1}+4^{-1}+8^{-1}+$...to 10 terms.

9. $\cdot 5+\cdot 05+\cdot 005+$......to $(n-1)$ terms.

10. $\frac{1}{10}+\frac{6}{100}+\frac{6}{1000}+\frac{6}{10000}+$......to n terms. [C. U. '11]

11. $3-6+12-$......-384. **12.** $1+\frac{1}{2}+\frac{1}{4}+$......$+\frac{1}{256}$.

Find the sum of the following without assuming any formula :—

13. $4+2+1+$......to 10 terms.

14. $2+4+8+$......to n terms.

15. $1+\frac{1}{3}+\frac{1}{3^2}+$......to n terms. [C. U. '12]

16. $6+12+24+$......$+768$.

17. The sum of the first and second terms of a G. P. is 12, and that of its fourth and fifth terms is 324. Find the sum of the first six terms of the series. [E. B. S. B. '49]

18. Find the sum of 25 terms of a G. P. whose 4th term is 20 and 7th term 160. [D. B. '45]

19. A mango tree yields every year twice as many mangoes as it did the previous year. If it yields 100 mangoes in the first year, how many mangoes will it yield in 8 years ?

20. Divide 21 into three parts such that they are in G. P. and their product is 64. [G. U. '53]

13. Miscellaneous Sums on Progression

Examples (10)

Ex. 1. If a, b, c, d are in G. P., show that a^2+b^2, b^2+c^2 and c^2+d^2 are also in G. P. [C. U. '19]

Here, a, b, c, d are in G. P. Let r be the common ratio.

$$\therefore \quad b=ar,\ c=ar^2,\ d=ar^3.$$

$$\therefore \quad a^2+b^2=a^2+a^2r^2=a^2(1+r^2),$$

$$b^2+c^2=a^2r^2+a^2r^4=a^2r^2(1+r^2),$$

$$c^2+d^2=a^2r^4+a^2r^6=a^2r^4(1+r^2).$$

Now, the terms $a^2(1+r^2)$, $a^2r^2(1+r^2)$, $a^2r^4(1+r^2)$ are in G. P., the common ratio being r^2.

$$\therefore \quad a^2+b^2,\ b^2+c^2,\ c^2+d^2 \text{ are in G. P.}$$

Ex. 2. If a, b, c are in G. P., prove that $\frac{1}{a+b}, \frac{1}{2b}, \frac{1}{b+c}$ are in A. P. [D. B. '46 ; G. P. '48]

Here a, b, c are in G. P. Let r be the commn ratio.

$$\therefore \quad b=ar,\ c=ar^2.$$

$\frac{1}{a+b}, \frac{1}{2b}, \frac{1}{b+c}$ will be in A. P., if we can prove that

$$\frac{1}{a+b}+\frac{1}{b+c}=\frac{1}{2b}\times 2=\frac{1}{b}.$$

$$\text{Now, } \frac{1}{a+b}+\frac{1}{b+c}=\frac{1}{a+ar}+\frac{1}{ar+ar^2}=\frac{1}{a(1+r)}+\frac{1}{ar(1+r)}$$

$$=\frac{r+1}{ar(1+r)}=\frac{1}{ar}=\frac{1}{b}.$$

Hence, $\frac{1}{a+b}, \frac{1}{2b}, \frac{1}{b+c}$ are in A. P.

Ex. 3. If a, b, c be in A. P. and a, b, d in G. P., show that $a, a-b, d-c$ are in G. P. [C. U. '10]

$\because \quad a, b, c$ are in A. P., $\therefore \quad a+c=2b$, or, $a-2b=-c \ldots(1)$

Again, $\because$ a, b, d are in G. P., $\therefore$ $b^2=ad\ldots(2)$.

$a, a-b, d-c$ will be in G. P., if it can be proved that

$$(a-b)^2=a(d-c)=ad-ac.$$

Now, $(a-b)^2=a^2-2ab+b^2=b^2+a(a-2b)$

$$=ad+a\times -c=ad-ac \quad [\text{ from (1) \& (2) }]$$

$\therefore$ $a, a-b, d-c$ are in G. P.

[This sum can be worked out as Ex. 1. and Ex. 2. above.]

Ex. 4. Insert between 6 and 16 two numbers such that the first three may be in A. P. and the last three in G. P.

Let b and c be the numbers.

Then from the given condition 6, b, c are in A. P....(1) and b, c, 16 are in G. P. ......(2).

From (1) we get $2b=6+c\ldots(3)$, and from (2) $c^2=16b\ldots(4)$.

Now, from (3) & (4) we have $c^2=8.2b=8(6+c)=48+8c$,

or, $c^2-8c-48=0$, or, $(c-12)(c+4)=0$, $\therefore$ $c=12$ or -4.

If $c=12$, then from (3) we get $b=9$,

If $c=-4$, „ „ (3) „ „ $b=1$.

$\therefore$ The reqd. numbers $=9$ and 12, or, 1 and -4.

Ex. 5. Insert 2 numbers between 5 and 135 so that the four may form a G. P. [C. U. '16]

The G. P. formed will have 4 terms, the first term being 5 and the fourth term 135. Let r be the common ratio.

$\therefore$ $135=t_4=5r^3$, or, $r^3=27=(3)^3$, $\therefore$ $r=3$.

$\therefore$ The reqd. numbers $=5\times 3$ and $5\times 3^2=15$ and 45.

Ex. 6. The sum of three quantities in G.P. is $24\frac{4}{5}$ and their product is 64 ; find them. [A. U. ; E. B. S. B. '50]

Let the quantities be $\frac{a}{r}$, a, ar.

$\therefore$ $\frac{a}{r}\times a\times ar=64\ldots(1)$ and $\frac{a}{r}+a+ar=24\frac{4}{5}\ldots(2)$.

Now from (1) we have $a^3=64$, $\therefore$ $a=4$.

From (2), we have $\frac{4}{r}+4+4r=\frac{124}{5}$,

or, $\frac{4}{r}+4r=\frac{124}{5}-4=\frac{104}{5}$, or, $\frac{1}{r}+r=\frac{26}{5}$,

or, $5r^2-26r+5=0$, or, $(r-5)(5r-1)=0$, $\therefore$ $r=5$ or $\frac{1}{5}$.

$\therefore$ The three numbers $=\frac{4}{5}, 4, 20$; or, $20, 4, \frac{4}{5}$.

Ex. 7. If a, b, c be respectively the pth, qth and rth terms of a G. P., prove that $a^{q-r}.\ b^{r-p}.\ c^{p-q}=1$.

[C. Pre-U. '63 ; C. U. '51 ; G. U. '50 ; W. B. S. F. '53]

Let f be the 1st term and d the common ratio.

$$\therefore\quad a=fd^{p-1},\ b=fd^{q-1} \text{ and } c=fd^{r-1}.$$

$$\therefore\quad a^{q-r}.\ b^{r-p}.\ c^{p-q}=(fd^{p-1})^{q-r}.\ (fd^{q-1})^{r-p}.\ (fd^{r-1})^{p-q}$$

$$=f^{q-r}.\ d^{pq-q-pr+r}\times f^{r-p}.d^{qr-r-pq+p}\times f^{p-q}.d^{rp-p-qr+q}$$

$$=f^{q-r+r-p+p-q}\times d^{pq-q-pr+r+qr-r-pq+p+rp-p-rq+q}$$

$$=f^0\times d^0=1\times 1=1.$$

Ex. 8. If a, b, c be in A. P., and x, y, z in G. P., prove that $x^{b-c}.\ y^{c-a}.\ z^{a-b}=1$. [C. U. '44, '50 ; G. U. '49]

$\because$ a, b, c are in A. P., $\therefore$ $2b=a+c$, or, $a-b=b-c$.

$\because$ x, y, z are in G. P., $\therefore$ $y^2=xz$.

Now, $x^{b-c}.y^{c-a}.z^{a-b}=x^{a-b}.y^{c-a}.z^{a-b}$ $[\because\ b-c=a-b]$

$$=(xz)^{a-b}.y^{c-a}=(y^2)^{a-b}.y^{c-a}=y^{2a-2b+c-a}$$

$$=y^{a+c-2b}=y^{2b-2b}=y^0=1.$$

Ex. 9. Sum to n terms the series $4+44+444$......

[Pat. U. '18]

Let s be the sum

$$s=4+44+444+\ldots\text{to } n \text{ terms}$$

$$=4(1+11+111+\ldots\text{to } n \text{ terms})$$

$$=\tfrac{4}{9}(9+99+999+\ldots\text{to } n \text{ terms})$$

$$=\tfrac{4}{9}\{(10-1)+(10^2-1)+(10^3-1)+\ldots\text{to } n \text{ terms}\}$$

$$=\tfrac{4}{9}\{(10+10^2+10^3+\ldots\text{to } n \text{ terms})-(n \text{ times } 1)\}$$

$$=\frac{4}{9}\left\{\frac{10(10^n-1)}{10-1}-n\right\}=\frac{40}{81}(10^n-1)-\frac{4n}{9}.$$

Ex. 10. Sum to n terms $\cdot 9+\cdot 99+\cdot 999+\ldots\ldots$

[C. U. (D. M. H.) '51)]

$$s=\cdot 9+\cdot 99+\cdot 999+\ldots\ldots\text{to } n \text{ terms}$$
$$=(1-\cdot 1)+(1-\cdot 01)+(1-\cdot 001)+\ldots\ldots\text{to } n \text{ terms}$$
$$=(n \text{ times } 1)-(\cdot 1+\cdot 01+\cdot 001+\ldots\ldots\text{to } n \text{ terms})$$
$$=n-\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\ldots\ldots\text{to } n \text{ terms}\right)$$
$$=n-\frac{\frac{1}{10}\left(1-\frac{1}{10^n}\right)}{1-\frac{1}{10}}=n-\frac{1}{9}\left(1-\frac{1}{10^n}\right).$$

Ex. 11. Sum $1+4+10+22+46+\ldots$to n terms.

Let s be the sum and t_n denote the nth term.

Then $s=1+4+10+22+46+\ldots+t_n$

Also $\underline{s=\quad 1+\ 4+10+22+\ldots+t_{n-1}+t_n}$

Subtracting, $0=(1+3+6+12+24+\ldots\text{to } n \text{ terms})-t_n$

$$\therefore\quad t_n=1+3+6+12+\ldots\text{to } n \text{ terms}$$
$$=1+\{3+6+12+\ldots\text{to } (n-1)^* \text{ terms}\}$$
$$=1+\frac{3(2^{n-1}-1)}{(2-1)}=1+3(2^{n-1}-1)=3.2^{n-1}-2.$$

Now putting 1, 2, 3,......, n for n we have

$$t_1=3.1-\ 2$$
$$t_2=3.2-\ 2$$
$$t_3=3.2^2-2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\underline{t_n=3.2^{n-1}-2}$$

$$\therefore\quad s=3(1+2+2^2+\ldots+2^{n-1})-2n$$

$$=3(1+2+2^2+\ldots\text{to } n \text{ terms})-2n=3\times\frac{1(2^n-1)}{2-1}-2n$$

$$=3.2^n-3-2n.$$

---

*Here, as 1 is not included in the G. P., the number of terms of the series $3+6+12\ldots$is $(n-1)$.

Ex. 12. Sum to n terms $1+\frac{3}{2}+\frac{7}{4}+\frac{15}{8}+\dots$ [C. U.]

$$s=1+\tfrac{3}{2}+\tfrac{7}{4}+\tfrac{15}{8}+\dots+t_n$$

$$\text{Also } s=\quad 1+\tfrac{3}{2}+\tfrac{7}{4}+\dots+t_{n-1}+t_n$$

$$\text{(Sub.) } 0=(1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\dots\text{to } n \text{ terms})-t_n$$

$$\therefore\quad t_n=1+\tfrac{1}{2}+\tfrac{1}{4}+\dots\text{to } n \text{ terms}=\frac{\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}=2\left(1-\frac{1}{2^n}\right)$$

Putting 1, 2, 3,...,n for n we get

$$t_1=2(1-\tfrac{1}{2})$$

$$t_2=2\left(1-\frac{1}{2^2}\right)$$

$$t_3=2\left(1-\frac{1}{2^3}\right)$$

....

$$t_n=2\left(1-\frac{1}{2^n}\right)$$

$$\therefore\quad s=2\left\{n-\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots+\frac{1}{2^n}\right)\right\}$$

$$=2\left\{n-\frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}\right\}=2\left\{n-1+\frac{1}{2^n}\right\}=2n-2+\frac{1}{2^{n-1}}.$$

Ex. 13. Sum $1+2a+3a^2+4a^3+\dots\dots$to n terms.

Here evidently $t_n=n.a^{n-1}$.

$$\therefore\quad s=1+2a+3a^2+4a^3+\dots+n.a^{n-1}$$

$$\text{and } sa=\quad a+2a^2+3a^3+\dots+(n-1)a^{n-1}+n.a^n$$

[Multiplying by a]

$$\text{(Sub.), } s(1-a)=(1+a+a^2+a^3+\dots+a^{n-1})-na^n$$

$$=\frac{1(1-a^n)}{1-a}-na^n.\qquad\therefore\quad s=\frac{1-a^n}{(1-a)^2}-\frac{na^n}{1-a}.$$

Ex. 14. Sum to n terms $(1)+(1+3)+(1+3+3^2)+(1+3+3^2+3^3)+\ldots\ldots$ [C. U. '31]

Here evidently $t_n=(1+3+3^2+3^3+\ldots\text{to } n \text{ terms})$

$$=\frac{1(3^n-1)}{3-1}=\frac{1}{2}(3^n-1)=\frac{1}{2}.\,3^n-\frac{1}{2}.$$

Now, putting 1, 2, 3,...,n for n we have

$$t_1=\tfrac{1}{2}.3^1-\tfrac{1}{2}$$
$$t_2=\tfrac{1}{2}.3^2-\tfrac{1}{2}$$
$$t_3=\tfrac{1}{2}.3^3-\tfrac{1}{2}$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$t_n=\tfrac{1}{2}.3^n-\tfrac{1}{2}$$

$$\therefore\ s=\frac{1}{2}(3+3^2+3^3+\ldots\ldots+3^n)-\frac{1}{2}n=\frac{1}{2}\times\frac{3(3^n-1)}{3-1}-\frac{n}{2}$$

$$=\frac{3}{4}(3^n-1)-\frac{n}{2}=\frac{3^{n+1}}{4}-\frac{3}{4}-\frac{n}{2}=\frac{1}{4}(3^{n+1}-3-2n).$$

Ex. 15. If a, b, c be in G. P., and x, y be the arithmetic means between a, b and b, c respectively. Prove that $\frac{a}{x}+\frac{c}{y}=2$ and $\frac{1}{x}+\frac{1}{y}=\frac{2}{b}$. [P. U. 1892]

$\because$ a, b, c are in G. P., $\therefore$ $b=ar, c=ar^2$ [r being *com. ratio*]

Again, $\because$ x and y are respectively the A. M.'s between a, b and between b, c,

$$\therefore\ x=\frac{a+b}{2} \text{ and } y=\frac{b+c}{2}.$$

Now, $\frac{a}{x}+\frac{c}{y}=\frac{a}{\frac{a+b}{2}}+\frac{c}{\frac{b+c}{2}}=2\left(\frac{a}{a+b}+\frac{c}{b+c}\right)=2\left(\frac{a}{a+ar}+\frac{ar^2}{ar+ar^2}\right)$

$$=2\left(\frac{ar+ar^2}{ar+ar^2}\right)=2.$$

Again, $\frac{1}{x}+\frac{1}{y}=\frac{1}{\frac{a+b}{2}}+\frac{1}{\frac{b+c}{2}}=2\left(\frac{1}{a+b}+\frac{1}{b+c}\right)$

$$=2\left(\frac{1}{a+ar}+\frac{1}{ar+ar^2}\right)=2\left(\frac{r+1}{ar+ar^2}\right)$$

$$=2\times\frac{r+1}{ar(r+1)}=\frac{2}{ar}=\frac{2}{b}.$$

Ex. 16. If S be the sum, P the product and R the sum of the reciprocals of n terms in G. P., prove that $\left(\frac{\mathrm{S}}{\mathrm{R}}\right)^n=\mathrm{P}^2$.

[C. Pre-U. ; B. U. E. '64 ; C. U. 1883]

Let a be the 1st term, r the common ratio, l the last term, and n the number of terms.

$$\therefore \quad \mathrm{S}=a+ar+\ldots+l=\frac{rl-a}{r-1}.$$

$\because$ The reciprocals of the terms are

$$\frac{1}{a},\ \frac{1}{ar},\ \frac{1}{ar^2},\ldots\ldots,\ \frac{1}{l}$$

$$\therefore \quad \mathrm{R}=\frac{\frac{1}{r}\times\frac{1}{l}-\frac{1}{a}}{\frac{1}{r}-1}=\frac{\frac{a-rl}{arl}}{\frac{1-r}{r}}=\frac{(a-rl)r}{(1-r)arl}=\frac{(rl-a)}{(r-1)al}.$$

$$\therefore \quad \left(\frac{\mathrm{S}}{\mathrm{R}}\right)^n=\left\{\frac{rl-a}{r-1}\times\frac{(r-1)al)}{(rl-a)}\right\}^n=(al)^n.$$

Now, $\mathrm{P}=a\times ar\times ar^2\times\cdots\times\frac{l}{r^2}\times\frac{l}{r}\times l$

Also, $\mathrm{P}=l\times\frac{l}{r}\times\frac{l}{r^2}\times\cdots\times ar^2\times ar\times a$ [writing in reverse order]

$$\therefore \quad \mathrm{P}^2=al\times al\times al\times\ldots\text{to } n \text{ factors}=(al)^n.$$

$$\therefore \quad \left(\frac{\mathrm{S}}{\mathrm{R}}\right)^n=\mathrm{P}^2.$$

Ex. 17. Show that the pth, qth, rth terms of a geometrical progression are in geometrical progression if p, q, r be in arithmetical progression. [W. B. S. F. 1952]

Let f be the 1st term and d the common ratio of the series.

$\therefore$ its pth term$=fd^{p-1}$, qth term$=fd^{q-1}$, and rth term $=fd^{r-1}$.

Now, fd^{p-1}, fd^{q-1}, and fd^{r-1} will be in G. P.,

if $(fd^{q-1})^2=fd^{p-1}.fd^{r-1}$, or, if $f^2d^{2q-2}=f^2d^{p+r-2}$,

or, if $d^{2q-2}=d^{p+r-2}$, or, if $2q-2=p+r-2$, or, if $2q=p+r$.

i.e., if p, q, r are in A. P.

Exercise 8

1. If x, y, z be in G. P., show that $x^2y^2z^2\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right)$
$=x^3+y^3+z^3$.

2. If a, b, c, d be in G. P., prove that

(*i*) a^2-b^2, b^2-c^2, c^2-d^2 are in G. P.

(*ii*) $a^2+b^2+c^2$, $ab+bc+cd$, $b^2+c^2+d^2$ are in G. P.

(*iii*) $(b-c)^2+(c-a)^2+(d-b)^2=(a-d)^2$.

[C. U. '43 ; D. B. '25, '26]

Find the sum of :—

3. $7+77+777+\ldots\ldots$to n terms.

4. $2+22+222+\ldots\ldots$to n terms.

5. $\cdot2+\cdot22+\cdot222+\ldots\ldots$to n terms.

6. $1.2+2.3+4.4+8.5+\ldots\ldots$to n terms.

7. $1+3+7+15+\ldots\ldots$to n terms. [D. B. '25, '26, '38]

8. $1+4+13+40+121+\ldots\ldots$to n terms.

9. If a be the first term of a G. P., l the nth term and P the product of first n terms, show that $P=(al)^{\frac{n}{2}}$.

[C. U. '18 ; D. B. '30, '33, '43, '47]

10. Sum to n terms the series of which the rth term is 2^r+2r. [D. B. '41]

11. If of three consecutive terms of a G. P. the middle term is 6 and the first and third terms are together equal to 15 ; find the series. [C. U. '32]

12. Sum to n terms the series $1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\ldots$

13. Three numbers whose sum is 15 are in A. P., if 1, 4 and 19 be added to them respectively the results are in G. P. Find the numbers. [N. B. Pr. '63 ; C. U. '50]

14. From three numbers in G. P. three other numbers in G. P. are subtracted and the remainders are found to be in G. P. ; prove that the three series must have the same common ratio. [B. U. 1890]

15. If s_1, s_2, s_3 be respectively the sums of n, $2n$ and $3n$ terms of a G. P., prove that $s_1(s_3-s_2)=(s_2-s_1)^2$. [B. U. 1882]

16. What must be added to x, y, z to bring them into G.P. ?

17. If G be the geometric mean, and M and N be two A.M.'s between two given quantities, show that $G^2=(2M-N)(2N-M)$.

[Hints : Let the numbers be a, b. $\therefore$ $G^2=ab$.

Again $\because$ a, M, N, b are in A. P., $\therefore$ a, M, N and also M, N, b are in A. P. $\therefore$ $2M=N+a$ and $2N=M+b\ldots$]

18. If the A.M. between p and q be twice their G.M., then $\frac{p}{q}=\frac{2+\sqrt{3}}{2-\sqrt{3}}$ or $\frac{2-\sqrt{3}}{2+\sqrt{3}}$.

$\Bigg[$ Hints : $\frac{p+q}{2}=\pm 2\sqrt{pq}$, or, $\frac{(p+q)^2}{4}=4pq$, or. $\frac{(p+q)^2}{4pq}=\frac{4}{1}\ldots(1)$

$$\therefore \quad \frac{(p+q)^2-4pq}{4pq}=\frac{3}{1} \text{ (by dividendo), or, } \frac{(p-q)^2}{4pq}=\frac{3}{1}\ldots(2).$$

Dividing (1) by (2) we have $\frac{(p+q)^2}{(p-q)^2}=\frac{4}{3}$, or, $\frac{p+q}{p-q}=\frac{2}{\sqrt{3}}$.

$$\therefore \quad \frac{2p}{2q}=\frac{2+\sqrt{3}}{2-\sqrt{3}} \text{ (by comp. and div.)}\ldots$$

If we write (2) as $\frac{(q-p)^2}{4pq}=3$, we may similarly have

$$\frac{p}{q}=\frac{2-\sqrt{3}}{2+\sqrt{3}} \Bigg]$$

19. Three numbers whose product is 512 are in G. P. ; if 8 be added to the first and 6 to the second, the resulting numbers are in A. P. Find the numbers. [C. U. (High) '50]

HARMONIC PROGRESSION

14. Definition. A series of quantities is said to be in **Harmonic Progression** if their reciprocals are in Arithmetical Progression.

Harmonic (or Harmonical) progression is briefly written as H. P.

Thus a, b, c are in H. P., if $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A. P.

So also $\frac{1}{5}, \frac{1}{7}, \frac{1}{9}$ are in H. P., for 5, 7, 9... are in A. P.

Corollary : Suppose a, b, c are three consecutive terms of an H. P.

Then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A. P.

$$\therefore \quad \frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}\ldots\text{(A)}, \text{ or, } \frac{a-b}{ab}=\frac{b-c}{bc}, \text{ or, } \frac{a-b}{b-c}=\frac{a}{c}\ldots(1)$$

Again, from (A) we have $\frac{2}{b}=\frac{1}{a}+\frac{1}{c}\ldots(2)$

and $\frac{2}{b}=\frac{1}{a}+\frac{1}{c}=\frac{a+c}{ac}, \quad \therefore \ b=\frac{2ac}{a+c}\ldots(3)$

Hence from (1) we may define that three quantities are in H. P., if the first term is to the third term as the difference of the first two terms is to the difference of the second and third terms.

N. B. If $a, a+b, a+2b\ldots$ be in A. P.,

then $\frac{1}{a}, \frac{1}{a+b}, \frac{1}{a+2b}, \ldots$ are in H. P.

Again since the nth term of $a, a+b, a+2b, \ldots$ is $a+(n-1)b$, the nth term of the H. P. $=\frac{1}{a+(n-1)b}$.]

HARMONIC MEANS

15. If three quantities be in harmonic progression, then the middle one is said to be the **harmonic mean** between the other two. Thus, if a, b, c be in H. P., then b is the harmonic mean between a and c.

Since a, b, c are in H. P., then $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are in A. P.

$$\therefore \quad \frac{1}{b}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{c}\right)=\frac{a+c}{2ac}, \quad \therefore \quad b=\frac{2ac}{a+b}.$$

Hence, we obtain here the relation between two terms and their harmonic mean.

[**N. B.** (1) $\because$ The reciprocals of the terms of an H. P. are in A. P., $\therefore$ to find any term (say nth term) of a series in H. P., we first find the nth term of the reciprocals of the terms of the series and the reciprocal of the nth term obtained is the nth term of the series in H. P.

(2) Similar is the process of finding n harmonic means between two given quantities.]

VARIATION

16. If the value of a quantity in an expression remains always the same, *i.e.*, does not vary with the changes in the values of other quantities in the expression, then this quantity is called a **constant** quantity (or a **constant**).

So the value of a constant quantity does not depend on the value of any other quantity. The quantity in an expression whose value varies is called a **variable** quantity or **variable.**

17. Variation. One quantity is said to **vary directly** as another, if the relation between the two is such that when the value of one of them is changed, the value of the other is also changed in the **same proportion.**

Examples : (i) Suppose a train runs uniformly at the rate of 20 miles an hour. If the time be doubled (*i.e.*, 2 hrs.), then the distance covered will also be doubled (*i. e.*, 40 miles). If the time be halved (*i.e.*, $\frac{1}{2}$ hr.), the distance will be halved (*i.e.*, 10 miles). Hence, it is seen here, that '*Distance covered is directly proportional to or varies directly as the time*'.

(ii) If r be the radius of a circle, its circumference is equal to $2\pi r$ (π is a constant here). If the radius be doubled, trebled etc, the circumference will also be doubled, trebled etc. Hence, it may be said that '*the circumference of a circle varies directly as its radius*'.

[**N. B.** Generally the word 'directly' is omitted. Thus, we say "x varies as y" and thereby mean that "x varies directly as y".

18. Variation symbol. The symbol of variation is $\propto$. Thus, to denote that 'x varies as y' we write $x \propto y$ (here $\propto$ denotes 'varies as').

19. Variation Constant. In the above example (*i*), suppose the train travels uniformly x miles in y hours.

As the train runs uniformly. the ratio $\frac{x}{y}$ will always remain the same. So if x varies as y, then $\dfrac{\text{any value of } x}{\text{corresponding value of } y}$ will always be the same. $\therefore$ $\frac{x}{y}$ is constant.

Suppose $\frac{x}{y} = k$ (k being a constant), $\therefore$ $x = ky$.

This constant k is called the constant of variation (or **variation constant**).

Note : If k is a constant and if $x = ky$, then $x \propto y$. Again, if $x = ky$, then $y = \frac{1}{k}x$ where $\frac{1}{k}$ is a constant quantity. Hence $y \propto x$. Different constants are used in different variations.

[**N. B.** Two quantities, which are so related that when one is increased or decreased, the other is also increased or decreased, are not necessarily proportional and hence one does not vary directly as the other.

For instance : If the **side** of a square is increased, its **area** is also increased, but the side and the area of a square are not proportional : for on doubling the side, we get 4 times the area, and on trebling the side, we get 9 times the area.]

We know that the graph of the equation $x=ky$ (where k is constant) is a straight line passing through the origin. So the ordinate of every point on this st. line is directly proportional to the abscissa.

Conversely, we may say that if the graph, drawn with the corresponding values of two variables as abscissæ and ordinates, passes through the orign, then one variable varies dirctly as the other.

20. The **value of the variation constant** can be found, if one pair of corresponding values of the variables be known.

Example 1. If x varies directly as y, and $x=7$ when $y=13$, find (i) the value of the variation constant, (ii) the relation between x and y and (iii) the value of x when $y=\frac{26}{35}$.

(i) $\because$ $x \propto y$, $\therefore$ $x=ky$ (k being variation constant).

Since $x=7$ when $y=13$,

$\therefore$ $7=k\times 13$, $\therefore$ $k=\frac{7}{13}$.

$\therefore$ the value of the variation constant $=\frac{7}{13}$.

(ii) $\because$ $k=\frac{7}{13}$, $\therefore$ the relation between x and y is $x=\frac{7}{13}y$.

(iii) Here $y=\frac{26}{35}$, $\therefore$ putting this value of y in $x=\frac{7}{13}y$, we have $x=\frac{7}{13}\times y=\frac{7}{13}\times\frac{26}{35}=\frac{2}{5}$.

$\therefore$ the reqd. value of $x=\frac{2}{5}$.

Example 2. The time of oscillation of a pendulum varies as the square root of its length. If a pendulum of length 40 inches oscillates once in a second, what is the length of the pendulum oscillating once in 2·5 seconds ?

Let t be the time for one oscillation and l be the length of the pendulum.

$\therefore$ From the given condition $t \propto \sqrt{l}$.

Let $t = k\sqrt{l}$ (k being variation constant).

$\because$ $t=1$ when $l=40$, $\therefore$ $1 = k\sqrt{40}$, or, $k = \dfrac{1}{\sqrt{40}}$.

$\therefore$ $t = \dfrac{1}{\sqrt{40}} \times \sqrt{l}$; putting $t = 2{\cdot}5$ in the equation, we have

$$2{\cdot}5 = \frac{\sqrt{l}}{\sqrt{40}}, \text{ or, } \frac{l}{40} = \frac{5}{2} \times \frac{5}{2}.$$

$\therefore$ $l = \frac{5}{2} \times \frac{5}{2} \times 40$ in. $= 250$ inches.

Reciprocal. If the product of two numbers be 1, then each number is the reciprocal of the other (*i.e.*, the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, the reciprocal of n is $\dfrac{1}{n}$, etc).

21. Inverse variation. One quantity is said to vary inversely as another, if the first varies directly as the **reciprocal** of the second.

Thus a varies inversely as b, if $a \propto \dfrac{1}{b}$ ($\because$ $\dfrac{1}{b}$ is the reciprocal of b).

$\therefore$ Here $a = k.\dfrac{1}{b}$ (k being variation constant), $\therefore$ $ab = k$.

Again, $\because$ $ab = k$, $\therefore$ $b = k.\dfrac{1}{a}$, so b also varies inversely as a.

Hence if one quantity varies inversely as another, the product of the two quantities is constant ; and conversely, if the product of two variable quantities is constant, each varies inversely as the other.

Instances : (1) When we travel a distance the **greater** the speed, the **less** will be the time required. If the speed be doubled, the time will be halved and if the speed be halved, the time required will be doubled. That is, if the speed be **multiplied** by any number, the time must be **divided** by the same number. Here, the speed and the time are in inverse variation.

(2) In rectangles of constant area, the lengths vary inversely as the breadths, *i.e.*, the bases vary inversely as the heights.

(3) If a certain number of men take a certain number of days to do a piece of work, then the double number of these men will do it in half the time, and so on.

Ex. 1. If $x \propto y$ and $a \propto \frac{1}{b}$, shew that $xa \propto \frac{y}{b}$.

$\because \; x \propto b, \; \therefore \; x = my$ (m being variation constant) ;

Again, $\because \; a \propto \frac{1}{b}, \; \therefore \; a = n.\frac{1}{b} = \frac{n}{b}$ (n being variation constant)

Now, $xa = my \times \frac{n}{b} = mn\frac{y}{b}$; $\because$ mn is a constant, $\therefore \; xa \propto \frac{y}{b}$.

[**N. B.** Here $\left(x \propto y \text{ and } a \propto \frac{1}{b}\right)$ in these two cases we have to take two different variation constants m and n.]

Ex. 2. If A is equal to the sum of two quantities one of which varies directly as B and the other inversely as B ; and if $A=7$ when $B=1$ and $A=10\frac{1}{3}$ when $B=3$, find the relation between A and B and the value of A when $B=2$.

Here A is the sum of two quantities, of which one varies directly as B and the other varies inversely as B. $\therefore$ The first quantity $=mB$ and the second $=\frac{n}{B}$ where m and n are constant.

$\therefore \quad A = mB + \frac{n}{B}$.

$\because \quad A=7$ when $B=1$, $\therefore \; 7 = m.1 + \frac{n}{1} = m + n$...(1)

Again, $\because \; A = 10\frac{1}{3}$, when $B=3$, $\therefore \; 10\frac{1}{3} = m.3 + \frac{n}{3}$,

or, $31 = 9m + n$...(2)

Solving (1) and (2) we have $m=3$ and $n=4$.

Hence, the relation between A and B is $A = 3B + \frac{4}{B}$.

Putting $B=2$ in this equation we have $A = 6 + \frac{4}{2} = 8$.

22. Graph. If x varies inversely as y, then x varies directly as $\frac{1}{y}$. If the points plotted with the corresponding values of x and $\frac{1}{y}$ lie in a st. line passing through the origin, i. e., if the graph, drawn taking the values of a variable as abscissæ and the corresponding values of the reciprocal of another variable as ordinates, be a st. line passing through the origin, then it may be said that the two variables are in inverse variation.

Again, here $xy=k$, and the graph of $xy=k$ is a rectangular hyperbola. Hence, if points are plotted with the corresponding values of two variable quantities, which are inversely proportional, as abscissæ and ordinates, then the points will lie on a rectangular hyperbola.

23. Joint variation. Sometimes we find that the value of one variable quantity depends on the values of more than one independent variables.

One quantity is said to vary jointly as a number of other quantities when it varies directly as their product.

Thus, if $A=kBC$ (k being a constant), then A varies jointly as B and C.

Instances : (1) The area of a triangle$=\frac{1}{2}b\times h$ (here $\frac{1}{2}$ is a constant, b the base and h the altitude of the triangle).

∴ The area of a triangle varies jointly as its base and altitude.

(2) Amount of work done varies jointly as the number of persons engaged and the number of days they work.

[**N. B.** (1) If x varies jointly as a, b, c, d,..., then $x=k\times abcd$...(where k is a constant).

Conversely, if $x=k\times abcd$...(k being constant), then x varies jointly as a, b, c, d, etc.

(2) If a quantity varies directly as a second and inversely as a third, then it varies jointly as the second and the reciprocal of the third. Thus, x varies directly as y and inversely as z if $x\propto\frac{y}{z}$, *i. e.*, if $x=m\frac{y}{z}$ (m being a constant).

For instance, the altitude of a triangle varies directly as the area and inversely as the base.

Example. If A varies as B and C jointly, and if $A=2$, when $B=\frac{3}{5}$, $C=\frac{10}{27}$; find C, when $A=54$ and $B=3$. [C. U. '20]

$\because$ $A\propto BC$, $\therefore$ $A=kBC$ (k being variation constant)...(1)

Putting $A=2$, $B=\frac{3}{5}$, $C=\frac{10}{27}$ in equation-(1),

we have $2=k\times\frac{3}{5}\times\frac{10}{27}=\frac{2}{9}k$, $\therefore$ $k=9$. $\therefore$ $A=9BC$...(2).

Now putting $A=54$ and $B=3$ in (2) we have

$54=9\times3\times C$, $\therefore$ $C=\frac{54}{27}=2$.

24. Theorem on joint variation. *If x varies as y when z is constant, and x varies as z when y is constant, then will x vary as the product of yz when both y and z vary.*

Proof. Here, the variation of x depends partly on the variation of y and partly on that of z. Suppose the variations of y and z take place separately, each in its turn producing its effect on x. Let a, b, c be certain simultaneous values of x, y, z respectively, *i. e.*, the original value of x becomes a when the variation is complete, and that of y becomes b and that of z becomes c.

(*i*) Let z be constant while y changes to b, at this stage x undergoes a partial change only dependent on y and will not change to a. Suppose x has the value a' on this partial change.

$$\therefore \quad \frac{x}{a'}=\frac{y}{b}\text{......(1).}$$

(*ii*) Let y be constant at its value b, while z changes to c. At this stage x completes its change, *i.e.*, its value changes from a' to a.

$$\therefore \quad \frac{a'}{a}=\frac{z}{c}\text{......(2).}$$

Now, from (1) and (2) we have $\frac{x}{a'}\times\frac{a'}{a}=\frac{y}{b}\times\frac{z}{c}$, or, $\frac{x}{a}=\frac{yz}{bc}$,

or, $x=\frac{a}{bc}\times yz$. Hence $x\propto yz$, since a, b, c are some fixed quantities.

[**N. B.** The theorem holds good, if there be more than two variables. If $A, B, C, D, \ldots$ be several quantities such that another quantity x varies directly with each of them while the others remain constant, then x will vary as their product when all of them vary.]

Instance : The area of a triangle varies as the base when the altitude is constant and the area varies as the altitude when the base is constant and hence the area varies as the product of the base and the altitude (*i.e.*, varies jointly as the base and altitude).

Cor. If $A \propto B$ when C is constant and $A \propto \frac{1}{C}$ when B is constant, then $A \propto \frac{B}{C}$ when both B and C vary.

Example : Apply the principle of variation to find how long 16 men will take to plough 25 acres, if 6 men take 10 days to plough 20 acres.

Let n be the number of men, d the number of days and A the number of acres. Evidently the number of men will vary directly as the number of acres, when number of days is constant, and if the number of acres is constant, the number of men will vary inversely as the number of days.

Hence $n \propto A$ when d is constant and $n \propto \frac{1}{d}$ when A is constant.

$\therefore \quad n \propto \frac{A}{d}$ when both A and d vary.

$\therefore \quad n = k. \frac{A}{d}$ (k being variation constant)...(1)

$\because$ From the given data $n = 6$ when $d = 10$ and $A = 20$,

$$\therefore \quad 6 = k. \frac{20}{10} = 2k, \quad \therefore \ k = 3.$$

Now, putting $A = 25$ and $n = 15$ in (1) we have

$15 = k. \frac{A}{d} = 3. \frac{25}{d}$, or, $15d = 75$, $\therefore \ d = 5$.

$\therefore$ the reqd. time $= 5$ days.

25. Some Elementary Properties.

4. If $A \propto B$, then $B \propto A$.

Proof. $\because$ $A \propto B$, $\therefore A = k.B$ (when k is variation constant).

$\therefore$ $B = \frac{1}{k}.A$, $\therefore$ $B \propto A$ ($\because$ $\frac{1}{k}$ is constant).

2. If $A \propto B$, then $A^m \propto B^m$.

Proof. $\because$ $A \propto B$, $\therefore$ $A = k.B$ (k being variation constant)

$\therefore$ $(A)^m = (kB)^m$, or, $A^m = k^m B^m$,

$\therefore$ $A^m \propto B^m$ ($\because$ k^m is constant).

3. If $A \propto B$ and $B \propto C$, then $A \propto C$.

Proof. $\because$ $A \propto B$, $\therefore$ $A = k.B$ (k being variation constant)

and $\because$ $B \propto C$, $\therefore$ $B = m.C$ (m being variation constant)

$\therefore$ $A = kB = k.mC = kmC$

$\therefore$ $A \propto C$ ($\because$ km is constant here).

4. If $A \propto BC$, then $B \propto \frac{A}{C}$ and $C \propto \frac{A}{B}$.

Proof. $\because A \propto BC$, $\therefore A = kBC$ (where k is variation constant)

$\therefore$ $B = \frac{1}{k}.\frac{A}{C}$, $\therefore$ $B \propto \frac{A}{C}$ ($\because$ here $\frac{1}{k}$ is constant)

Similarly, $C = \frac{1}{k}.\frac{A}{B}$, $\therefore$ $C \propto \frac{A}{B}$ ($\because$ $\frac{1}{k}$ is a constant)

5. If $A \propto B$, then $AC \infty BC$ (where C is either constant or variable).

Proof. $\because$ $A \infty B$, $\therefore$ $A = k$B (k being variation constant).

$\therefore$ $AC = k.BC$, $\therefore$ $AC \infty BC$ ($\because$ k is constant).

6. If $A \propto C$ and $B \propto C$, then $(A \pm B) \propto C$ and $AB \propto C^2$.

Proof. $\because$ $A \propto C$, $\therefore$ $A = m.C$ (where m is constant)...(1)

and $\because$ $B \propto C$, $\therefore$ $B = n.C$ (n being constant) ...(2)

Now, taking the sum and difference of (1) and (2)

we have $A \pm B = (m \pm n)C$,

$\therefore$ $(A \pm B) \propto C$ (Since m, n being constants both $m+n$ and $m-n$ are constant).

Again, $AB = mC \times nC = mn.C^2$.

$\therefore$ $AB \propto C^2$ ($\because$ mn is constant).

7. If $A \propto B$ and $C \propto D$, then $AC \propto BD$, and $\frac{A}{C} \propto \frac{B}{D}$.

[C. U. '23]

Proof. $\because A \propto B, \therefore A = m\mathrm{B}$ (m being variation constant)...(1)

and $\because C \propto D. \therefore C = nD$ (n being variation constant)...(1)

$\therefore$ from $(1) \times (2)$ we get $AC = mnBD$,

$\therefore \mathrm{AC} \infty \mathrm{BD}$ ($\because mn$ is constant).

Again from $(1) \div (2)$ we have $\frac{\mathrm{A}}{\mathrm{C}} = \frac{m}{n} . \frac{\mathrm{B}}{\mathrm{D}}$,

$\therefore \frac{\mathrm{A}}{\mathrm{C}} \propto \frac{\mathrm{B}}{\mathrm{D}}$ ($\because \frac{m}{n}$ is also constant).

Examples (11)

Ex. 1. If A varies as B and also as C, shew that A varies as $B - C$.

[C. U. '25]

$\because A \propto B, \therefore A = kB$ (k being variation constant), $\therefore B = \frac{A}{k}$...(1)

Again, $\because A \infty C, \therefore A = mC$ (m is variation constant), $\therefore C = \frac{A}{m}$...(2)

Subtracting (2) from (1) we have

$$B - C = \frac{A}{k} - \frac{A}{m} = A\left(\frac{1}{k} - \frac{1}{m}\right) = A\left(\frac{m-k}{km}\right), \ \therefore \ A = \frac{km}{m-k}(B - C).$$

Hence, $A \propto B - C$ (here $\frac{km}{m-k}$ is constant, k and m being constant).

Ex. 2. If $\frac{a}{b} \propto a + b$ and $\frac{b}{a} \propto a - b$, shew that $a^2 - b^2$ is invariable.

$\because \frac{a}{b} \propto a + b, \ \therefore \ \frac{a}{b} = k(a + b)$ where k is variation constant ;

Again, $\because \frac{b}{a} \propto a-b, \quad \therefore \quad \frac{b}{a}=m(a-b)$ where m is a constant ;

$$\therefore \quad \frac{a}{b} \times \frac{b}{a}=km(a+b)(a-b)$$

$$\text{or,} \quad 1=km(a^2-b^2), \quad \therefore \quad a^2-b^2=\frac{1}{km}=\text{a constant.}$$

$\therefore \quad a^2-b^2$ is invariable.

Ex. 3. Complete the following :—

(i) If $x \propto a^2$, then $a \propto$ ......

(ii) If $x \propto \frac{1}{\sqrt{a}}$, then $a \propto$ ......

(i) $\because x \propto a^2, \quad \therefore \quad x=ka^2$ (where k is variation constant)

$$\therefore \quad a^2=\frac{x}{k}, \quad \therefore \quad a=\frac{\sqrt{x}}{\sqrt{k}}.$$

$\because \quad \sqrt{k}$ is constant, $\quad \therefore \quad a \propto \sqrt{x}$.

$\therefore$ If $x \propto a^2$, then $a \propto \sqrt{x}$.

(ii) $\because \quad x \propto \frac{1}{\sqrt{a}}, \quad \therefore \quad x=k.\frac{1}{\sqrt{a}}$ (k being variation constant)

or, $x^2=\frac{k^2}{a}, \quad \therefore \quad a=\frac{k^2}{x^2}=k^2. \ \frac{1}{x^2}$.

$\because \quad k^2$ is constant, $\quad \because \quad a \propto \frac{1}{x^2}$.

$\therefore$ If $x \propto \frac{1}{\sqrt{a}}$, then $a \propto \frac{1}{x^2}$.

Ex. 4. If $A \propto B^2$ and if $B=4$ when $A=4$, find B when $A=3$ and also find A when $B=9$. Find also B in terms of A.

$\because \quad A \propto B^2, \quad \therefore \quad A=kB^2$ (where k is a constant)

Putting the given values of A and B in $A=kB^2$ we get

$4=k.(4)^2, \quad \therefore \quad k=\frac{1}{4}. \quad \therefore \quad A=\frac{1}{4}B^2$.

Now, if $A=3$, then $3=\frac{1}{4}B^2$, or, $B^2=12, \quad \therefore \quad B=\pm 2\sqrt{3}$.

Again, putting $B=9$ we have $A=\frac{1}{4} \times 9^2=\frac{81}{4}=20\frac{1}{4}$.

Again, $\because \quad kB^2=A, \quad \therefore \quad \frac{1}{4}B^2=A$, or, $B^2=4A$,

$$\therefore \quad B=\pm \sqrt{4A}=\pm 2\sqrt{A}.$$

Ex. 5. If A^2+B^2 varies as A^2-B^2, shew that A varies as B. [C. U. '23]

$\because \quad A^2+B^2 \propto A^2-B^2$,

$\therefore \quad A^2+B^2=k(A^2-B^2)$ [where k is variation constant],

$\therefore \quad \dfrac{A^2+B^2}{A^2-B^2}=\dfrac{k}{1}$, whence by comp. & divd. we have

$$\frac{A^2+B^2+A^2-B^2}{A^2+B^2-A^2+B^2}=\frac{k+1}{k-1}, \text{ or, } \frac{2A^2}{2B^2}=\frac{k+1}{k-1},$$

or, $\dfrac{A^2}{B^2}=\dfrac{k+1}{k-1}$, or, $A^2=\dfrac{k+1}{k-1}.B^2$,

or, $A=\sqrt{\dfrac{k+1}{k-1}}.B$, $\therefore A \propto B \left(\because \sqrt{\dfrac{k+1}{k-1}} \text{ is constant}\right)$.

Ex. 6. Given that $x+y$ varies as $z+\frac{1}{z}$ and that $x-y$ varies as $z-\frac{1}{z}$; find the relation between x and z provided that $z=2$ when $x=3$ and $y=1$. [P. U. '48]

$\because \quad x+y \propto z+\dfrac{1}{z}$,

$\therefore \quad x+y=k\left(z+\dfrac{1}{z}\right)$, where k is variation constant...(1)

Again $\because \quad x-y \propto \left(z-\dfrac{1}{z}\right)$,

$\therefore \quad x-y=m\left(z-\dfrac{1}{z}\right)$, where m is variation sonstant...(2)

Adding (1) and (2) we have $2x=k\left(z+\dfrac{1}{z}\right)+m\left(z-\dfrac{1}{z}\right)$......(3).

Now, since $z=2$ when $x=3$, $y=1$,

$\therefore$ from (1), $3+1=k(2+\frac{1}{2})$, or, $\frac{5}{2}k=4$, $\therefore \quad k=\frac{8}{5}$.

Again, from (2), $3-1=m(2-\frac{1}{2})$, or, $\frac{3}{2}m=2$, $\therefore \quad m=\frac{4}{3}$.

$\therefore$ from (3) we have $2x=\dfrac{8}{5}\left(z+\dfrac{1}{z}\right)+\dfrac{4}{3}\left(z-\dfrac{1}{z}\right)$,

or, $2x=(\frac{8}{5}+\frac{4}{3})z+(\frac{8}{5}-\frac{4}{3})\dfrac{1}{z}=\frac{44}{15}z+\dfrac{4}{15z}$, or, $x=\frac{22}{15}z+\dfrac{2}{15z}$,

or, $15x=22z+\dfrac{2}{z}$ which is the required relation between x and z.

Ex. 7. If $a \propto b^2$ and $1+b \propto \sqrt{c}$, find a in terms of c if $c=9$ and $b=5$ when $a=1$.

$\because$ $a \propto b^2$, $\therefore$ $a=mb^2$ (here m is variation constant)...(1).
Again, $\because$ $1+b \propto \sqrt{c}$, $\therefore$ $1+b=n\sqrt{c}$ (here n is a constant)...(2)
Putting, the values of a and b in (1) we have $1=m(5)^2$,

$$\therefore \quad m=\tfrac{1}{25}.$$

Again, putting the values of b and c in (2) we have
$1+5=n\sqrt{9}=3n$, $\therefore$ $n=2$.
Now, from (2) we have $1+b=2\sqrt{c}$ ($\because$ $n=2$),

$$\text{or,}\quad b=2\sqrt{c}-1, \quad \therefore \quad b^2=4c-4\sqrt{c}+1.$$

Hence from (1) we have $a=mb^2$, or, $a=\dfrac{b^2}{25}$ ($\because$ $m=\tfrac{1}{25}$)

$$\therefore \quad a=\frac{4c-4\sqrt{c}+1}{25}.$$

Ex. 8. If x varies directly as the square of y and inversely as the cube root of z, and if $x=2$, when $y=4$, $z=8$, find the value of y, when $x=3$ and $z=27$. [C. U. '17]

$$\because \quad x \propto y^2 \quad \text{and} \quad x \propto \frac{1}{\sqrt[3]{z}},$$

$$\therefore \quad x \propto y^2 \cdot \frac{1}{\sqrt[3]{z}}, \quad \therefore x=k\frac{y^2}{\sqrt[3]{z}} \text{ (here } k \text{ is variation constant).}$$

Putting $x=2$, $y=4$ and $z=8$ we have

$$2=k.\frac{4^2}{\sqrt[3]{8}}=k.\tfrac{16}{2}=8k, \quad \therefore \quad k=\tfrac{1}{4}.$$

$\therefore$ $x=\dfrac{1}{4}\cdot\dfrac{y^2}{\sqrt[3]{z}}$, now putting $x=3$ and $z=27$ in this we have

$$3=\frac{1}{4}\cdot\frac{y^2}{\sqrt[3]{27}}=\frac{y^2}{4\times 3}, \text{ or, } \quad y^2=36, \quad \therefore \quad y=\pm 6.$$

Ex. 9. If $b \propto a^3$, find the ratio in which b is increased if a is increased in the ratio 3 : 2.

If a is increased in the ratio 3 : 2, the value of a^3 increases in the ratio 27 : 8.

Now, $\because$ $b \propto a^3$, $\therefore$ b will be increased in the ratio 27 : 8 if a is increased in the ratio 3 : 2.

Ex. 10. If x varies inversely as y, show that $x+y$ is least when $x=y$.

$\because \quad x \propto \frac{1}{y}, \quad \therefore \quad x=k.\frac{1}{y}$, or, $xy=k$ (where k is a constant)

Now, $(x+y)^2=(x-y)^2+4xy=(x-y)^2+4k$

Here $4k$ is constant,

$\therefore$ the value of $(x+y)^2$ will be the least when $(x-y)^2=0$.

Hence, if $(x-y)^2=0$, *i.e.*, if $x=y$, then $x+y$ will be the least.

Ex. 11. If $x+y \propto x-y$, show that (i) $x^2+y^2 \propto xy$,

(ii) $ax+by \propto px+py$, a, b, p, q being all constants.

[C. U. '36, P. U. '47]

(i) $\because \quad x+y \propto x-y, \quad \therefore \quad x+y=k(x-y)$, where k is constant :

$\therefore \quad (x+y)^2=k^2(x-y)^2$ [squaring],

or, $x^2+y^2+2xy=k^2(x^2+y^2)-2k^2xy$,

or, $2xy+2k^2xy=k^2(x^2+y^2)-(x^2+y^2)$ [by transposition]

or, $2(k^2+1)xy=(k^2-1)(x^2+y^2)$

$$\therefore \quad x^2+y^2=\frac{2(k^2+1)}{k^2-1}.xy$$

$\therefore \quad x^2+y^2 \propto xy$ (since k is constant).

(ii) $\because \quad x+y \propto x-y, \quad \therefore \quad x+y=k(x-y)$ where k is a constant,

$\therefore \quad \frac{x+y}{x-y}=k, \quad \therefore$ by comp. & div. we have

$$\frac{x}{y}=\frac{k+1}{k-1}=m \text{ (say), } \quad \therefore \quad x=my.$$

$$\text{Now, } \frac{ax+by}{px+qy}=\frac{amy+by}{pmy+qy}=\frac{y(am+b)}{y(pm+q)}=\frac{am+b}{pm+q},$$

$$\therefore \quad ax+by=\frac{am+b}{pm+q}(px+qy).$$

$\because$ a, b, p, q are constants and m is also a constant k being a constant, $\therefore$ it is proved that $ax+by \propto px+qy$.

Ex. 12. If $x+y\propto z$ when y is constant and if $z+x\propto y$ when z is constant, show that when both y and z vary, then $x+y+z\propto yz$. [B. U. E. '64 : G. U. '49]

$\because$ $x+y\propto z$ (when y is constant)

$\therefore$ $x+y=kz$ (k being variation constant)

$\therefore$ $x+y+z=kz+z=(k+1)z$; hence $k+1$ being constant

$x+y+z\propto z$ (when y is constant).

Again, $\because$ $x+z\propto y$ (when z is constant)

$\therefore$ $x+z=my$ (here m is variation constant)

$\therefore$ $x+z+y=my+y=(m+1)y$.

$\therefore$ $m+1$ being constant $x+y+z\propto y$ (when z is constant).

Hence, from the theorem of joint variation we have

$x+y+z\propto yz$, when both y and z vary.

Ex. 13. If x, y, z be variable quantities such that $y+z-x$ is constant and $(x+y-z)(x+z-y)\propto yz$, prove that $x+y+z\propto yz$. [P. U. '40]

Let $y+z-x=k$ (a constant).

$\because$ $(x+y-z)(x+z-y)\propto yz$,

$\therefore$ $(x+y-z)(x+z-y)=myz$ (where m is a constant),

or, $x^2-(y-z)^2=myz$,

or, $x^2-(y-z)^2-4yz=myz-4yz$,

or, $x^2-(y+z)^2=(m-4)yz$,

or, $(x+y+z)(x-y-z)=(m-4)yz$

or, $(x+y+z)\times(-k)=(m-4)yz$ [$\because$ $y+z-x=k$],

$\therefore$ $x+y+z=\dfrac{m-4}{-k}.yz=\dfrac{4-m}{k}.\,yz$;

Hence, $x+y+z\propto yz$ ($\because$ $\dfrac{4-m}{k}$ is constant m and k being constants.)

Ex. 14. Two globes of gold that have their radii equal to r and r' are melted and formed into a single globe ; find its radius (the volume of a globe varies as the cube of the radius).

[C. U. '31]

Let V and v be the volumes of the globes. Their radii are r and r'.

$\because$ $V \propto r^3$, $\therefore$ $V = mr^3$ (where m is the variation constant)

Again, $\because$ $v \propto r'^3$, $\therefore$ $v = mr'^3$ (m being variation constant)

$\therefore$ $V + v = m(r^3 + r'^3)$.

Now the volume of the single globe is $V + v$. Let R be its radius.

$\therefore$ $V + v = mR^3$ (m being the variation constant)

Hence, $mR^3 = m(r^3 + r'^3)$, $\therefore$ $R = \sqrt[3]{r^3 + r'^3}$.

Ex. 15. The time of oscillation of a pendulum varies as the square root of its length. If a pendulum of length 8 feet oscillates once in 3·1 seconds, find the time for a pendulum 10 ft. long.

Let t be the time for one oscillation and l feet be the length of the pendulum.

$\therefore$ From the given condition $t \propto \sqrt{l}$,

$\therefore$ $t = k\sqrt{l}$ (where k is variation constant),

$\because$ $t = 3{\cdot}1$ secs., when $l = 8$ ft.,

$$\therefore \quad 3{\cdot}1 = k\sqrt{8} = k.2\sqrt{2}, \quad \therefore \quad k = \frac{3{\cdot}1}{2\sqrt{2}}.$$

$\therefore$ We have $t = \dfrac{3{\cdot}1}{2\sqrt{2}}\sqrt{l}$. Now putting $l = 10$ in this equn.,

we have $t = \dfrac{3{\cdot}1}{2\sqrt{2}} \times \sqrt{10} = \dfrac{3{\cdot}1 \times \sqrt{20}}{4} = \dfrac{3{\cdot}1 \times 2\sqrt{5}}{4} = 3{\cdot}5$ (nearly).

$\therefore$ The reqd. time $= 3{\cdot}5$ seconds (App.).

Ex. 16. The mass m of a body varies as density d when the volume v is constant and varies as the volume v when density d

is constant. If unit mass be defined as mass of a body of unit volume and unit density, show that $m=vd$. [C. U. '26]

$\because$ $m \propto d$ when v is constant and $m \propto v$ when d is constant,

$\therefore$ $m \propto vd$, $\therefore$ $m=kvd$ (where k is variation constant)...(1)

$\therefore$ from the given data $m=1$, when $v=1$ and $d=1$,

$\therefore$ from (1) we have $1=k \times 1 \times 1=k$.

Hence, from (1) we have $m=kvd=vd$ [$\because$ $k=1$].

Ex. 17. A playground, whose length and width are in the ratio 8 : 7, has two-thirds of it reserved for accommodation. If the width is to be diminished by one-ninth, in what ratio should the length be increased in order that the accommodation may be trebled ? [C. U. '32]

Let x and y be respectively the length and width of the play ground. $\therefore$ $\frac{2}{3}xy$ is reserved for accommodation and $\frac{1}{3}xy$ for the play ground.

$\because$ The area of the ground reserved for play is a constant,

$\therefore$ to treble the accommodation, the new area of the whole ground should be $(\frac{2}{3}xy \times 3+\frac{1}{3}xy)$ or $\frac{7}{3}xy$. If the width y is to be diminished by one-ninth, the width becomes $(y-\frac{1}{9}y)$ or $\frac{8}{9}y$.

$\therefore$ The length becomes $(\frac{7}{3}xy \div \frac{8}{9}y)$ or $\frac{21}{8}x$.

Hence, the length should be increased in the ratio 8 : 21.

Exercise 9

1. If $b \propto a^3$ and $b=50$ when $a=5$, find b when $a=1$ and find a when $b=6\frac{1}{4}$. Find also b in terms of a.

2. If $y \propto \frac{1}{x^2}$ and if $y=9$ when $x=10$, find y if $x=6$ and find x if $y=4$. Find also y in terms of x.

3. If A varies as B and C jointly and if $A=2$ when $B=\frac{3}{5}$ and $C=\frac{10}{27}$, find C when $A=54$ and $B=3$. [C. U. '20]

4. If $a^2+b^2 \propto ab$, show that $a+b \propto a-b$.

5. The resistance (R) to the motion of a train of given weight is partly constant and partly varies as the square of the velocity (v). Express the statement by symbols.

6. Complete the following :—

(*i*) If $a \propto b^3$, $b \propto \ldots$

(*ii*) If $t \propto \sqrt{l}$, $l \propto \ldots.$

✓**7.** If $a \propto b$ and $b \propto c$, show that $a^3+b^3+c^3 \propto 3abc$.

8. If x varies directly as y and inversely as z, and $x=\frac{1}{6}$ when $y=5$ and $z=9$, find the relation between x, y and z. Hence find the value of x when $y=6$ and $z=\frac{1}{5}$.

✓**9.** If $x-y \propto z$ when y is constant and $x-z \propto y$ when z is constant, show that $x-y-z \propto yz$ when y and z both vary.

10. If b is equal to the sum of two quantities one of which varies directly as a, and the other inversely as a, and if $b=5$ when $a=1$ and $b=12.5$ when $a=6$, find the relation between a and b. Find the value of b when $a=3$.

11. A varies as the sum of two other quantities, one of which varies directly as B^2 and the other inversely as C. If $A=16$ when $B=2$ and $C=1$, and if $A=5$ when $B=1$ and $C=2$, find the value of A when $B^2=3$ and $C^2=16$.

✓**12.** $x \propto \frac{1}{y^2}$, find the ratio in which x is increased if y is increased in the ratio 7 : 4.

13. The area of a circle varies as the square of its radius ; if the area is $17\frac{1}{9}$ sq.ft. when the radius is 2 ft. 4 in., find the area when the radius is 3 ft. 6 in.

14. If x vary directly as the square of y and inversely as the cube root of z, and if $x=2$ when $y=4$ and $z=8$, find the value of y when $x=3$ and $z=27$. [C. U. '37]

✓**15.** Apply the principle of variation to find how long 25 men will take to plough 30 acres, if 5 men take 9 days to plough 10 acres of land. [C. U. '34]

16. The length of a pendulum varies inversely as the square of the number of beats it makes per minute. If a pendulum 16 ft. long makes 27 beats per minute, find the length of the pendulum that makes 24 beats per minute.

17. If $x \propto yz^2$, $y \propto ab^2$ and $z \propto \frac{b}{a}$, find how x varies with a, b.

18. If in the variation $x=ky$ and $y=k'z$, a, b, c and a', b', c' be two sets of values of x, y, z,

show that $\dfrac{a^2+b^2+c^2}{aa'+bb'+cc'}=\dfrac{aa'+bb'+cc'}{a'^2+b'^2+c'^2}$. [C. U. '22]

19. If a stone falls s ft. in t seconds from rest, $s \propto t^2$. If it is observed to fall 64 ft. in 2 secs., find how far it falls in 4 seconds.

20. The pressure of wind on a plane surface varies jointly as the area of the surface and the square of the wind's velocity. If the pressure on a square foot is 1 lb. when the wind's velocity is 16 miles per hour, find the velocity of the wind when the pressure on a square yard is $14\frac{1}{16}$ lb.

21. Pressure $(=p)$ in a liquid varies as depth $(=d)$ when the density $(=\mathrm{D})$ is constant and it varies as density when depth is constant. The pressure is 1 when the depth is 32 and the density 1. Find the depth at which the pressure is 2 when the density is 16. [C. U. '21]

22. The electrical resistance of a wire is proportional directly to its length and inversely to the square of its diameter. Compare the resistance of two wires of the same material, one of which has a diameter of 1·5 *mm.* and is 4 *m.* long while the other has a diameter of 2 *mm.* and is 5*m.* long.

23. The volume of a sphere varies as the cube of the radius and the surface of a sphere varies as the square of the radius Show that the square of the volume varies as the cube of the surface. [C. U. 1924]

24. The cost of a dinner is partly constant and partly varies as the number of guests. If the cost is Rs. 275 for 150 guests and Rs. 320 for 240 guests ; find the cost for 250 guests.

25. The cost of boring a well, f feet deep, partly varies as f and partly as f^2. Such a well costs Rs. 130 if the depth is 40 ft. and costs Rs. 255 if the depth is 60 ft. How deep is the well if the cost is Rs. 420 ?

✓**26.** In a certain machine a force of P pounds will support a load of W pounds and it is known that P is partly constant and partly proportional to W. If $P=14$ when $W=44$ and $P=26$ when $W=92$, draw a graph to show the value of P for any load between 40 lbs. and 100 lbs. Find the value (*i*) of P when $W=76$, and (*ii*) of W when $P=20$.

LOGARITHM

26. Definition. The **Logarithm** of any real number with respect to a given **base** is the **index of the power** to which the base is to be raised in order to be equal to the given number.

We know $2^3=8$, here 2 is the **base** and 3 is the **index** of the power of 2, *i.e.*, 3 is the index of the power to which the base 2 is to be raised to produce 8. Hence 3 is the logarithm of 8 to the base 2.

It is briefly written as $3=\log_2 8$. Similarly $\because\ 3^2=9$. $\therefore\ 2=\log_3 9$. Generally, if $a^x=\text{M}$, then $x=\log_a \text{M}$. Conversely, if $x=\log_a \text{M}$, then $a^x=\text{M}$.

27. Some properties of Logarithms.

(*a*) $\log_a(\text{MN})=\log_a\text{M}+\log_a\text{N}$;

(*b*) $\log_a\dfrac{\text{M}}{\text{N}}=\log_a\text{M}-\log_a\text{N}$;

(*c*) $\log_a\text{M}^n=n\log_a\text{M}$;

(*d*) $\log_a\text{M}=\log_b\text{M}\times\log_a b$;

(*e*) $\log_a 1=0$: (*f*) $\log_a a=1$.

Proofs : (*a*) Let $x=\log_a \mathrm{M}$ and $y=\log_a \mathrm{N}$.

Then $a^x=\mathrm{M}$ ...(1) and $a^y=\mathrm{N}$......(2).

Multiplying (1) by (2) we have $a^x.a^y=\mathrm{MN}$,

or, $a^{x+y}=\mathrm{MN}$, $\therefore$ $\log_a(\mathrm{MN})=x+y=\log_a\mathrm{M}+\log_a\mathrm{N}$.

Cor. $\log_a(xyz...)=\log_a x+\log_a y+\log_a z+...$. This is true for any number of factors.

(*b*) Let $x=\log_a\mathrm{M}$ and $y=\log_a\mathrm{N}$.

Then $a^x=\mathrm{M}$...(1) and $a^y=\mathrm{N}$...(2).

Dividing (1) by (2) we have $\frac{a^x}{a^y}=\frac{\mathrm{M}}{\mathrm{N}}$, or, $a^{x-y}=\frac{\mathrm{M}}{\mathrm{N}}$,

$$\therefore \quad \log_a \frac{\mathrm{M}}{\mathrm{N}}=x-y=\log_a\mathrm{M}-\log_a\mathrm{N}.$$

Cor. $\log_a \frac{xyz...}{mnp...}=\log_a x+\log_a y+\log_a z+...-\log_a m-\log_a n$
$-\log_a p-...$

(*c*) $\log_a\mathrm{M}^n=\log_a$ (M.M.M......to n factors)
$=\log_a\mathrm{M}+\log_a\mathrm{M}+\log_a\mathrm{M}+$......to n terms
$=n\log_a\mathrm{M}$.

(*d*) Let $x=\log_a\mathrm{M}$ and $y=\log_b\mathrm{M}$, then $a^x=\mathrm{M}$ and $b^y=\mathrm{M}$.

$\therefore$ $a^x=b^y$, or, $a^{\frac{x}{y}}=b$, $\therefore$ $\frac{x}{y}=\log_a b$, or, $x=y\log_a b$,

$\therefore$ $\log_a\mathrm{M}=\log_b\mathrm{M}\times\log_a b$.

[**N. B.** Here the base has been changed from *a* to *b*. Thus to change the base from *a* to *b*, we have to divide the given logarithm by $\log_a b$.]

Cor. Putting $\mathrm{M}=a$ in the above formula,

we have $\log_b a\times\log_a b=1$.

(*e*) We know $a^0=1$, $\therefore$ $\log_a 1=0$.

Hence, logarithm of 1 to any base (excepting 0) is 0.

(*f*) $\because$ $a^1=a$, $\therefore$ $\log_a a=1$.

Hence, the logarithm of any number to itself as base is 1.

[**N. B.** The logarithm of 0 to a base (say *a*) less than 1 is $+\infty$ and that of 0 to a base greater than 1 is $-\infty$.]

(i) Suppose the mantissa of the logarithm of 37 is to be found. We find from the table that the number noted in the row to the right of 37 and under the head 0 is 5682. So the mantissa of log 37 is ·5682.

(ii) To find the mantissa of log 374, we have to note the number in the row to the right of 37 and under the head 4. It is found to be 5729. Hence the mantissa of log 374 is ·5729.

(iii) The mantissa of the logarithm of 3746, a number of 4 digits :

As in (ii) we find the mantissa of log 374 to be ·5729. The fourth digit of the given number is 6, for which we have to move farther to the right to find the figure tabled under the head 6 in the second section of small types. The number 7 is noted there which denotes ·0007. So the difference for 6 is ·0007. Hence the mantissa of log 3746 is (·5729 + ·0007) or ·5736.

N. B. The characteristic of the logarithm of 3746 is 3 (i.e., one less than the number of digits in 3746) and its mantissa is ·5736.

Hence log $3746 = 3{\cdot}5736$. Similarly, log $37 = 1{\cdot}5682$ and log $374 = 2{\cdot}5729$.

34. Mantissæ of logarithms of numbers having the same digits.

The logarithms of numbers consisting of the same significant digits arranged in the same order but differing only in the position of the decimal point have the same mantissa.

We have already found log $3746 = 3{\cdot}5736$.

Now, $\log 374{\cdot}6 = \log \frac{3746}{10} = \log 3746 - \log 10$

$= 3{\cdot}5736 - 1 = 2{\cdot}5736$;

$\log 37{\cdot}46 = \log \frac{3746}{100} = \log 3746 - \log 100 = 3{\cdot}5736 - 2 = 1{\cdot}5736$;

$\log 3{\cdot}746 = \log \frac{3746}{1000} = \log 3746 - \log 1000 = 3{\cdot}5736 - 3 = {\cdot}5736$;

$$\log {}^{\cdot}3746 = \log \tfrac{3746}{10000} = \log 3746 - \log 10000 = 3{}^{\cdot}5736 - 4$$
$$= 3 + {}^{\cdot}5736 - 4 = {}^{\cdot}5736 - 1 = \bar{1}{}^{\cdot}5736.$$

[**N. B.** log ˙3746=˙5736 − 1 which is equal to − ˙4264, but as the common logarithm is always written with decimal part (mantissa) positive, log ˙3746 is written as $\bar{1}$˙5736 and not as − ˙4264. In $\bar{1}$˙5736, only the characteristic (*i.e.*, $\bar{1}$) is negative, but the mantissa (˙5736) is positive. To denote that only the characteristic is negative a *bar* is put over 1 here. It is read as *bar 1*. Thus $\bar{2}$ is read as bar 2 etc. In $\bar{1}$, $\bar{2}$ etc. 1, 2 etc. are negative. If we write − 1˙5736 it will mean that the entire number 1˙5736 is negative.

35. Antilogarithm or Antilog.

If the logarithm of the numbor *m* is *n*, then *m* is the anti-logarithm (antilog) of *n*. So the number corresponding to a given logarithm is called its **antilogarithm.**

Tnus log 37˙46=1˙5736.

∴ antilogarithm of 1˙5736 is 37˙46.

36. To find the antilogarithm of a given logarithm.

See the **Antilog Table** at the end of this book. In the first column of this table numbers with two places of decimals are given one below the other. These are the first two digits in the mantissa of a logarithm.

(1) Suppose the antilogarithm of 1˙5736 is to be determined. Its mantissa is ˙5736. We find the number 3741 tabled under the head 3 in the horizontal line to the right of ˙57. Farther to the right in the same line we find the number 5 tabled under the head 6. Thus for the mantissa ˙5736 we have (3741+5) or 3746. Hence ˙5736 is the mantissa of the logarithm of a number whose significant digits are 3746. Now, since 1 is the characteristic here, there must be two figures in the integral part of the number obtained above from the table.

Hence antilog 1·5736 = 37·46. Similarly antilog 2·5736 = 374·6 and antilog $\bar{2}$·5736 = ·03746.

(*i*) Find the antilogarithm of − ·5378.

Here − ·5378 = − 1 + 1 − ·5378 = − 1 + ·4622 = $\bar{1}$·4622.

∴ the antilogarithm of − ·5378 is the antilogarithm of $\bar{1}$·4622. Here ·4622 is the mantissa of the logarithm of the number whose significant figures are 2898 as found from the antilog table. Now the position of the decimal point in the number we seek is determined by the characteristic $\bar{1}$ here,

∴ antilog $\bar{1}$·4622 = ·2898.

Examples (12)

Ex. 1. Find the logarithm of (*a*) 78, (*b*) 324, (*c*) 1·362 and (*d*) ·035.

(*a*) 78 consists of two digits, so the characteristic of its logarithm will be 1. To find the mantissa of log 78 from the table we take the mantissa of 780 and from the table the corresponding significant digits are 8921. So the mantissa is ·8921.

∴ log 78 = 1·8921.

(*b*) As 324 consists of three digits the characteristic of its log will be 2. Now, from the Table the mantissa of the logarithm of 324 is found to be ·5105.

∴ log 324 = 2·5105.

(*c*) As the integral part of 1·362 consists of 1 digit, the characteristic of its log is 0.

From the log table we find 1335 under the head 6 in the horizontal line with 13. Again farther in the same line under the head 2 we find the number 7. 1335 + 7 = 1342,

∴ log 1·362 = ·1342.

(*d*) In ·035 there is one zero after the decimal point and hence the characteristic of its log is − 2 *i.e.*, $\bar{2}$.

From the table we have the mantissa of log 35 = ·5441.

∴ log ·035 = $\bar{2}$·5441.

Ex. 2. (*a*) Subtract (*i*) 2·5732 and (*ii*) $\bar{4}{\cdot}2714$ from the sum of $\bar{4}{\cdot}7654$ and 1·4351.

$$\begin{array}{r} \text{Sum} \\ \bar{4}{\cdot}7654 \\ \underline{1{\cdot}4351} \\ \bar{2}{\cdot}2005 \end{array}$$

Here after adding the decimal figures we have to carry 1. Hence the characteritic is the sum of -4, 1 and 1 *i.e.*, -2 which is written as $\bar{2}$.

$$\begin{array}{rr} \text{Now } (i) & \bar{2}{\cdot}2005 \\ & \underline{2{\cdot}5732} \\ & \bar{5}{\cdot}6273 \end{array}$$

When we come to the integral part (in the subtraction), we have to subtract $(2+1)$ or 3 from -2. The result is -5 which is written as $\bar{5}$.

$$\begin{array}{rr} (ii) & \bar{2}{\cdot}2005 \\ & \underline{\bar{4}{\cdot}2714} \\ & 1{\cdot}9291 \end{array}$$

Here in the integral part $(-4+1)$ or -3 is to be subtracted from -2. The result is 1.

(*b*) (*i*) Multiply $\bar{2}{\cdot}4265$ by 5 and (*ii*) divide $\bar{3}{\cdot}5304$ by 4.

$$\begin{array}{rr} (i) & \bar{2}{\cdot}4265 \\ & \underline{\times 5} \\ & \bar{8}{\cdot}1325 \end{array}$$

Multiplying 4 by 5 in the decimal part we have to carry 2 to the product of -2 and 5. ∴ The characteristic is $-10+2$ or -8 written as $\bar{8}$.

$$\begin{array}{rl} (iii) & \bar{3}{\cdot}5304 \div 4 \\ & = \frac{1}{4}(\bar{3}{\cdot}5304) \\ & = \frac{1}{4}(\bar{4}+1{\cdot}5304) \\ & = \bar{1}+{\cdot}3826 \\ & = \bar{1}{\cdot}3826 \end{array}$$

In dividing a logarithm with a negative characteristic we have to write the characteristic so that its negative part is divisible by the divisor. Here $\bar{3}$ is not divisible by 4. Hence $\bar{3}$ is written as $\bar{4}+1$.

Ex. 3. Using table find the antilog of (*a*) $\bar{1}{\cdot}2463$ and (*b*) $-2{\cdot}8254$.

(*a*) Here the characteristic is $\bar{1}$ and the mantissa is ·2463. From the antilog table we find that ·2463 is the mantissa of the logarithm of the number whose significant figures are 1763. Now, the position of the decimal point in the required number is determined by the characteristic $\bar{1}$, which shows there is no integral part in the number and the first figure after the decimal point is a significant figure. ∴ The reqd. antilog = ·1763.

(*b*) Here $-2{\cdot}8254=-3+1-{\cdot}8254=-3+{\cdot}1746=\bar{3}{\cdot}1746$.

From the antilog table we find that ·1746 is the mantissa of the logarithm of the number whose significant figures are 1495. Here the charactristic is $\bar{3}$, so there will be two zeroes after the decimal point in the required number.

$\therefore$ the reqd. antilog$={\cdot}001495$.

Ex. 4. Using log tables find the value of $\dfrac{1}{(1{\cdot}045)^{20}}$.

[P. U. '50]

Let $x=\dfrac{1}{(1{\cdot}045)^{20}}$.

$$\therefore \log x=\log\frac{1}{(1{\cdot}045)^{20}}=\log 1-\log(1{\cdot}045)^{20}$$
$$=\log 1-20\log 1{\cdot}045=0-20\times{\cdot}0191=-{\cdot}382$$
$$=-1+1-{\cdot}382=-1+{\cdot}618=\log{\cdot}4150$$

(from antilog table)

$\therefore$ $x={\cdot}415$. $\therefore$ the reqd. value$={\cdot}415$.

Ex. 5. Using the log tables find the value of $\dfrac{\sqrt[5]{2{\cdot}415}}{(0{\cdot}824)^4}$.

[P. U. 1948]

Let $x=\dfrac{\sqrt[5]{2{\cdot}415}}{(0{\cdot}824)^4}$, $\therefore$ $\log x=\log\dfrac{\sqrt[5]{2{\cdot}415}}{(0{\cdot}824)^4}$

$$=\log(2{\cdot}415)^{\frac{1}{5}}-\log({\cdot}824)^4$$
$$=\tfrac{1}{5}\log 2{\cdot}415-4\log{\cdot}824$$
$$=\tfrac{1}{5}\times 0{\cdot}3829-4\times\bar{1}{\cdot}9159$$
$$={\cdot}0765+4-3{\cdot}6636={\cdot}4129$$
$$=\log 2{\cdot}587 \text{ (from antilog table)}$$

$\therefore$ $x=2{\cdot}587$. $\therefore$ the reqd. value$=2{\cdot}587$.

Ex. 6. If $\log x=2{\cdot}5785$, find x.

Here the characteristic of the given log is 2.

$\therefore$ the integral part in the value of x consists of 3 digits.

Now ·5785 is the mantissa of the logarithm of a number whose significant figures are 3788 (from table).

$\therefore$ $2{\cdot}5785=\log 378{\cdot}8$. $\therefore$ $x=378{\cdot}8$.

Ex. 7. Find the value of $(1{\cdot}035)^{-16}$ from the log table.

Let $x=(1{\cdot}035)^{-16}$

$$\therefore \quad \log x=\log(1{\cdot}035)^{-16}=-16\log 1{\cdot}035=-16\times{\cdot}0149$$
$$=-{\cdot}2384=-1+1-{\cdot}2384$$
$$=-1+{\cdot}7616=\bar{1}{\cdot}7616$$
$$=\log {\cdot}5776 \text{ (from antilog table)}$$
$$\therefore \quad x={\cdot}5776, \quad \therefore \text{ the reqd. value}={\cdot}5776.$$

Ex. 8. Find the number of digits in $(6)^{25}$, having given $\log 2={\cdot}3010$ and $\log 3={\cdot}4771$.

$$\log(6)^{25}=25\log 6=25\log(2\times 3)=25(\log 2+\log 3)$$
$$=25({\cdot}3010+{\cdot}4771)=25\times{\cdot}7781=19{\cdot}4525.$$

$\because$ the characteristic of $\log(6)^{25}$ is 19,

$\therefore$ $(6)^{25}$ consists of 20 digits.

Ex. 9. Find the position of the first significant figure in the value of 2^{-30}; given that $\log 2={\cdot}3010$.

Let $x=2^{-30}$

$$\therefore \quad \log x=\log 2^{-30}=-30\times\log 2=-30\times{\cdot}3010$$
$$=-9{\cdot}03=-10+1-{\cdot}03=\overline{10}{\cdot}97$$

$\because$ the characteristic of $\log x$ is $\overline{10}$,

$\therefore$ there are $(10-1)$ or 9 zeroes after the decimal point.

$\therefore$ the first significant figure occupies the 10th position.

Ex. 10. Find from the log tables the value of

$$\frac{3{\cdot}78\times{\cdot}032\times 109{\cdot}2\times 19{\cdot}895}{{\cdot}00078\times 981}.$$

log of the numerators	log of the denominators
$\log\ 3{\cdot}78={\cdot}5775$	$\log\ {\cdot}00078=\bar{4}{\cdot}8921$
$\log\ {\cdot}032=\bar{2}{\cdot}5051$	$\log\ 981=2{\cdot}9917$
$\log\ 109{\cdot}2=2{\cdot}0382$	$\bar{1}{\cdot}8838$
$\log\ 19{\cdot}895=1{\cdot}2987$	
$2{\cdot}4195$	

Now $2{\cdot}4195-(\bar{1}{\cdot}8838)=2{\cdot}4195+1-{\cdot}8838=2{\cdot}5357$

$\because$ Antilog $2{\cdot}5357=343{\cdot}4$,

$\therefore$ the reqd. value $=343{\cdot}4$.

Ex. 11. Find the square root of ·0265.

$$\sqrt{\cdot 0265}=(\cdot 0265)^{\frac{1}{2}}.\quad \text{Now } \log(\cdot 0265)^{\frac{1}{2}}=\tfrac{1}{2}\log\cdot 0265$$
$$=\tfrac{1}{2}\times\bar{2}\cdot 4232=\bar{1}\cdot 2116=\log\cdot 1628$$

$\therefore\ (\cdot 0265)^{\frac{1}{2}}=\cdot 1628.\ \therefore$ the reqd. square root $=\cdot 1628$.

Ex. 12. If $\log 2=\cdot 30103$, find $\log 5$.

$\log 5=\log\frac{10}{2}=\log 10-\log 2=1-\cdot 30103=\cdot 69897$.

Ex. 13. If $\log 2=\cdot 3010300$ and $\log 13=1\cdot 1139434$, find $\log 65$.

$$\log 65=\log(5\times 13)=\log 5+\log 13=\log\tfrac{10}{2}+\log 13$$
$$=\log 10-\log 2+\log 13=1-\cdot 3010300+1\cdot 1139434$$
$$=1\cdot 8129134.$$

Ex. 14. Find the value of $\sqrt[5]{35\cdot 28}$, given that $\log 2=\cdot 3010$, $\log 3=\cdot 4771$, $\log 7=\cdot 8451$ and $\log 203\cdot 9=2\cdot 3095$.

$35\cdot 28=\frac{3528}{100}=\frac{2^3\times 3^2\times 7^2}{10^2}$; let $x=\sqrt[5]{35\cdot 28}$,

$$\therefore\ \log x=\log(35\cdot 28)^{\frac{1}{5}}=\log\left(\frac{2^3\times 3^2\times 7^2}{10^2}\right)^{\frac{1}{5}}$$
$$=\tfrac{1}{5}\,[3\log 2+2\log 3+2\log 7-2\log 10]$$
$$=\tfrac{1}{5}\,[3\times\cdot 3010+2\times\cdot 4771+2\times\cdot 8451-2\times 1]$$
$$=\tfrac{1}{5}\,[\cdot 9030+\cdot 9542+1\cdot 6902-2]$$
$$=\tfrac{1}{5}\times 1\cdot 5474=\cdot 3095 \text{ [to 4 places of decimals]}$$

Now $\because\ \log 203\cdot 9=2\cdot 3095,\ \therefore\ \log 2\cdot 039=\cdot 3095,$

$\therefore\ x=2\cdot 039.\ \therefore\ \sqrt[5]{35\cdot 28}=2\cdot 039.$

Ex. 15. Find the value of 1·231 of £ 2. 14s. 7d. to the nearest penny.

The given exp. $=655$ d. $\times 1\cdot 231=x$ (suppose)

$\therefore\ \log x=\log 655+\log 1\cdot 231$

$=2\cdot 8162+\cdot 0902$ (from log table)

$=2\cdot 9064=\log 806\cdot 1$ (from Antilog table)

$\therefore\ x=806\cdot 1$; $\therefore$ the reqd. value $=806\cdot 1$ d. $=$ £ 3. 7s.2d. (App.)

Ex. 16. Find (to the nearest rupee) the amount at compound interest on Rs. 2150 for 3 yrs. at 5%.

The amount $=\text{principal}\left(1+\frac{\text{rate p. c.}}{100}\right)^{\text{yr}}$.

$$=2150\times(1+\tfrac{5}{100})^3=2150\times(1{\cdot}05)^3.$$

Now, $\log\{2150\times(1{\cdot}05)^3\}=\log 2150+3\log 1{\cdot}05$

$=3{\cdot}3324+3\times{\cdot}0212=3{\cdot}3960=\log 2489.$

$\therefore$ the reqd. amount $=$ Rs. 2489.

Ex. 17. In a right-angled triangle, the sides containing the right angle are 27·4 inches and 32·5 inches. Find its area.

Area of the $\triangle=\frac{1}{2}\times 27{\cdot}4\times 32{\cdot}5$ sq. in. $=13{\cdot}7\times 32{\cdot}5$ sq. in.

Now, $\log(13{\cdot}7\times 32{\cdot}5)=\log 13{\cdot}7+\log 32{\cdot}5$

$=1{\cdot}1367+1{\cdot}5119=2{\cdot}6486=\log 445{\cdot}2$

$\therefore$ the reqd. area $=445{\cdot}2$ sq. inches.

Ex. 18. The population of a town is 3000. If it increases annually at the rate of 10%, what will be the population at the end of 3 yrs.?

The reqd. population $=3000\times(1+\frac{10}{100})^3=3000\times(1{\cdot}1)^3$

Now, $\log\{3000\times(1{\cdot}1)^3\}=\log 3000+3\log 1{\cdot}1$

$=3{\cdot}4771+3\times{\cdot}0414=3{\cdot}6013=\log 3993.$

$\therefore$ the reqd. population $=3993$.

Ex. 19. Prove that $\log\frac{65}{14}+\log\frac{77}{143}=\log 5-\log 2$.

$\log\frac{65}{14}=\log 65-\log 14=\log(5\times 13)-\log(2\times 7)$

$=\log 5+\log 13-\log 2-\log 7;$

and $\log\frac{77}{143}=\log 77-\log 143=\log 7+\log 11-\log 11-\log 13$

$\therefore\ \log\frac{65}{14}+\log\frac{77}{143}=\log 5+\log 13-\log 2-\log 7$

$+\log 7+\log 11-\log 11-\log 13$

$=\log 5-\log 2.$

Ex. 28. If $\log(x^3y^2)=3a+2b$ and $\log(x^2y^3)=2a+3b$, find $\log x$ and $\log y$ in terms of a and b. [C. U. 48]

$$\because\ \log x^3y^2=3a+2b,$$

$$\therefore\ 3\log x+2\log y=3a+2b\ldots\ldots(1)$$

Again, $\because\ \log x^2y^3=2a+3b,$

$$\therefore\ 2\log x+3\log y=2a+3b\ldots\ldots(2)$$

Now solving (1) and (2) we have $\log x=a$ and $\log y=b$.

Exercise 10

1. Find the logarithms of 88, ·1234 and ·003256.

2. Find the antilogs of 1·362, $-1{\cdot}2463$ and $\bar{2}{\cdot}8254$.

3. If $\log 2={\cdot}30103$ and $\log 3={\cdot}47712$, find the number of digits in 2^{17} and 3^{20}.

4. Find the position of the first significant digit in the value of 3^{-13}, given that $\log 3={\cdot}47712$.

5. Find the square root of 2·7 ane the 5th root of 8·216 and the square of $\frac{7}{8}$.

6. Find from the log table the value of :—

(i) $\sqrt[3]{45{\cdot}37}\times({\cdot}7692)^4$. [P. U. '49]

(ii) $\dfrac{32{\cdot}47\times4{\cdot}8\times873{\cdot}2}{523{\cdot}4\times{\cdot}04236\times391}$ (iii) $\dfrac{42{\cdot}36\times3{\cdot}802\times{\cdot}524}{578{\cdot}3\times8{\cdot}129}$

7. Given $\log 2={\cdot}3010300$ and $\log 3={\cdot}4771213$, find the value of log 45. [C. U. '51]

8. If $\log 2={\cdot}30103$, find the value of $7\log\frac{10}{9}-2\log\frac{25}{24}+3\log\frac{81}{80}$.

9. Find the value of £ 13. 6*s.* 8*d.* × 5·25.

10. Find the price of 196 articles at Rs. 4. 5*a.* 2*p.* each.

11. Find the mean proportional between 55 and 1375.

12. Find in acres the area of a rectangular field 2253 yds. by 269 yds.

13. Find (in nearest rupees) the amount from Rs. 425 in 3 yrs. at 4% compound interest.

14. If the present population of a town be 6000 and if it increases annually at the rate of 5%, what will be the population in 2 years ?

15. Find the radius of the circle whose area is $\frac{11}{35}$ acres.

16. Find log 324 when the base is $3\sqrt{2}$.

17. Prove that $7 \log \frac{16}{15} + 5 \log \frac{25}{24} + 3 \log \frac{81}{80} = \log 2$. [C. U. '36]

18. Show that $\log_{10} 2 + 16 \log_{10} \frac{16}{15} + 12 \log_{10} \frac{25}{24} + 7 \log_{10} \frac{81}{80} = 1$. [C. U. '40]

19. If 1 metre = 39·37 inches, express 1 sq. metre in sq. yds.

20. If $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$, shew that $xyz = 1$.
[H. S. '61 (Compl.)]

21. If three numbers be in G. P., show that their logarithms are in A.P.

22. Solve : (i) $2^x.3^{2x} = 100$ [C. U. '25]
(ii) $7^{3x+2} + 4^{x+2} = 7^{3x+1} + 2^{2x+6}$
by the help of log tables. [C. U. '41]

23. Find the logarithm of 1728 to the base $2\sqrt{3}$.

24. Show that $\log \frac{b^n}{c^n} + \log \frac{c^n}{a^n} + \log \frac{a^n}{b^n} = 0$. [C. U. '44]

25. If a, b, c, d be any 4 positive numbers, prove that
$\log_b a \times \log_c b \times \log_d c = \log_d a$. [C. U. '42]

26. If $\log (x^2y^3) = a$ and $\log \left(\frac{x}{y}\right) = b$, find $\log x$ and $\log y$ in terms of a and b. [C. U. '19]

27. If $\log_a b = 10$ and $\log_{6a}(32b) = 5$, find a. [C. U. '49]

28. If $a^{3-x}b^{5x} = a^{x+5}b^{3x}$, show that $x \log \left(\frac{b}{a}\right) = \log a$.
[C. U. '37]

Solve the following equations by the help of log-tables when required :—

29. $6^{3-4x}4^{x+5} = 8$. [C. U. '45]

30. $2^x.7^y = 80000$, $3^y = 500$. [C. U. '47]

31. $5^{5-3x} + 4^{\frac{1}{2}x+3} = 5^{7-3x} - 2^{x+5}$ (correct to 3 places of decimals). [C. U. '43]

IRRATIONAL QUANTITIES

37. Quantities that can be expressed as the ratio of two integers are called **rational** quantities ; and the quantities that cannot be so expressed are called **irrational** quantities.

These have been dealt with in the previous class (vide "surds" in part I).

38. Some Properties of Binomial surds.

I. (*a*) The product and the quotient of two similar quadratic surds are rational.

Proof. Let $a\sqrt{x}$ and $b\sqrt{x}$ be two similar quadratic surds, Their product$=a\sqrt{x}\times b\sqrt{x}=abx$, which is rational ;

their quotient$=\dfrac{a\sqrt{x}}{b\sqrt{x}}=\dfrac{a}{b}$, which is rational.

(*b*) Conversely, if the product or quotient of two surds be rational, then they are similar.

Proof. Let the product of two surds $\sqrt{x}$ and $\sqrt{y}$ be a rational quantity p.

$$\therefore\quad \sqrt{x}\times\sqrt{y}=p,\quad \therefore\quad \sqrt{x}=\frac{p}{\sqrt{y}}=\frac{p}{y}.\sqrt{y}$$

$$=\text{rational quantity}\left(\frac{p}{y}\right)\times\sqrt{y}.$$

Hence $\sqrt{x}$ and $\sqrt{y}$ are similar.

Similarly it can be proved in the case of quotient also.

Cor. The product and quotient of two unlike quadratic surds are irrational.

II. A quadratic surd cannot be equal to the sum or difference of a rational quantity and a quadratic surd.

Proof. If possible let $\sqrt{a}=b\pm\sqrt{c}$.

By squaring we get $a=b^2+c\pm2b\sqrt{c}$,

$$\therefore\quad \sqrt{c}=\pm\frac{a-b^2-c}{2b},\text{ which is rational.}$$

Thus here a surd is equal to a rational quantity, which is absurd.

Hence the supposition being absurd, the theorem is proved.

III. If $x+\sqrt{y}=a+\sqrt{b}$, where x and a are rational and $\sqrt{y}$ and $\sqrt{b}$ are irrational, then $x=a$ and $y=b$.

Proof. If x and a are not equal, let $x=a+m$.

Now, $a+\sqrt{b}=x+\sqrt{y}=a+m+\sqrt{y}$,

$\therefore\ \sqrt{b}=m+\sqrt{y}$, *i. e.* a quadratic surd is equal to the sum of a rational quantity and a surd, which is absurd.

$\therefore\ x=a$. Hence $\sqrt{y}=\sqrt{b}$, $\therefore\ y=b$.

Cor. Similarly it can be proved that if $x-\sqrt{y}=a-\sqrt{b}$, then $x=a$ and $y=b$.

[**N. B.** It is evident from (III) that in equations of the forms $x\pm\sqrt{y}=a\pm\sqrt{b}$, the rational parts of the two sides are equal and so also are the two irrational parts of the two sides.

So $x+\sqrt{y}=a+\sqrt{b}$ is equivalent to two equations $x=a$ and $\sqrt{y}=\sqrt{b}$ or $y=b$, provided $\sqrt{y}$ and $\sqrt{b}$ are real irrational quantities.]

IV. If $\sqrt{(x+\sqrt{y})}=\sqrt{a}+\sqrt{b}$, then $\sqrt{(x-\sqrt{y})}=\sqrt{a}-\sqrt{b}$.

Proof. $\because\ \sqrt{(x+\sqrt{y})}=\sqrt{a}+\sqrt{b}$,

$\therefore$ Squaring both sides we get $x+\sqrt{y}=a+b+2\sqrt{ab}$.

$\therefore\ x=a+b$ and $\sqrt{y}=2\sqrt{ab}$.

$\therefore\ x-\sqrt{y}=a+b-2\sqrt{ab}=(\sqrt{a}-\sqrt{b})^2$,

$\therefore\ \sqrt{(x-\sqrt{y})}=\sqrt{a}-\sqrt{b}$.

Cor. If $\sqrt{(x-\sqrt{y})}=\sqrt{a}-\sqrt{b}$, then $\sqrt{(x+\sqrt{y})}=\sqrt{a}+\sqrt{b}$.

V. If $\sqrt[3]{(x+\sqrt{y})}=a+\sqrt{b}$, then $\sqrt[3]{(x-\sqrt{y})}=a-\sqrt{b}$.

Proof. $\because\ \sqrt[3]{(x+\sqrt{y})}=a+\sqrt{b}$, $\therefore$ cubing both sides we have $x+\sqrt{y}=a^3+3a^2\sqrt{b}+3ab+b\sqrt{b}$ [$\because\ (\sqrt{b})^3=(\sqrt{b})^2.\sqrt{b}$ $=b\sqrt{b}$]

$$=(a^3+3ab)+(3a^2+b)\sqrt{b}.$$

$$\left.\begin{aligned}\therefore\ x&=a^3+3ab\ldots\ldots(1)\\ \text{and }\sqrt{y}&=(3a^2+b)\sqrt{b}\ldots(2)\end{aligned}\right\}$$ Equating the rational parts and the irrational parts of the two sides.

Now, subtracting (2) from (1) we have

$x-\sqrt{y}=a^3-3a^2\sqrt{b}+3ab-b\sqrt{b}=(a-\sqrt{b})^3,$

$\therefore\ \sqrt[3]{(x-\sqrt{y})}=a-\sqrt{b}.$

Cor. If $\sqrt[3]{(x-\sqrt{y})}=a-\sqrt{b}$, then $\sqrt[3]{(x+\sqrt{y})}=a+\sqrt{b}$.

39. Square root of Quadratic Surds

We know that the square of the expression $\sqrt{a}+\sqrt{b}$ is the sum of a rational quantity (here, $a+b$) and a quadratic surd (here, $2\sqrt{ab}$).

So, the square of $\sqrt{a}+\sqrt{b}$ can be expressed in the form $x+\sqrt{y}$. Hence the square root of expressions like $x+\sqrt{y}$ will be in the form $\sqrt{a}+\sqrt{b}$.

Ex. 1. Find the square root of $x+\sqrt{y}$, where $\sqrt{y}$ is a quadratic surd.

Let $\sqrt{(x+\sqrt{y})}=\sqrt{a}+\sqrt{b}$. Squaring both sides we get $x+\sqrt{y}=a+b+2\sqrt{ab}$.

$\therefore\ x=a+b\ldots(1)$ and $2\sqrt{ab}=\sqrt{y}$, or, $4ab=y\ldots(2)$

$\therefore\ (a-b)^2=(a+b)^2-4ab=x^2-y$

$\therefore\ a-b=\sqrt{x^2-y}\ldots(3)$

$\therefore$ Adding (1) and (3) we have $a=\frac{1}{2}(x+\sqrt{x^2-y})$, and subtracting (3) from (1) we get $b=\frac{1}{2}(x-\sqrt{x^2-y})$.

$\therefore$ The required square root

$=\pm[\sqrt{\{\frac{1}{2}(x+\sqrt{x^2-y})\}}+\sqrt{\{\frac{1}{2}(x-\sqrt{x^2-y})\}}].$

[Similarly to find the square root of $x-\sqrt{y}$, put $\sqrt{(x-\sqrt{y})}=\sqrt{a}-\sqrt{b}$.]

Ex. 2. Find the square root of $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$.

If the given expression has a square root it must be of the form $\sqrt{x}+\sqrt{y}+\sqrt{z}$.

So, let $\sqrt{(a+\sqrt{b}+\sqrt{c}+\sqrt{d})}=\sqrt{x}+\sqrt{y}+\sqrt{z}$.

Squaring both sides we have

$a+\sqrt{b}+\sqrt{c}+\sqrt{d}=x+y+z+2\sqrt{xy}+2\sqrt{yz}+2\sqrt{zx}.$

$\therefore$ Here $a=x+y+z$, $\sqrt{b}=2\sqrt{xy}$, $\sqrt{c}=2\sqrt{yz}$,

and $\sqrt{d}=2\sqrt{zx}$.

Now $\sqrt{b}\times\sqrt{c}=2\sqrt{xy}\times 2\sqrt{yz}$, ∴ $\sqrt{bc}=4y\sqrt{xz}$,

∴ $\frac{\sqrt{bc}}{\sqrt{d}}=2y$ [∵ $\sqrt{d}=2\sqrt{xz}$], ∴ $y=\frac{1}{2}\sqrt{\frac{bc}{d}}$.

Similarly we have $x=\frac{1}{2}\sqrt{\frac{bd}{c}}$ and $z=\frac{1}{2}\sqrt{\frac{cd}{b}}$.

∴ the required square root

$$=\pm\left\{\sqrt{\left(\frac{1}{2}\sqrt{\frac{bd}{c}}\right)}+\sqrt{\left(\frac{1}{2}\sqrt{\frac{bc}{d}}\right)}+\sqrt{\left(\frac{1}{2}\sqrt{\frac{cd}{b}}\right)}\right\}.$$

[**N. B.** This square root can be found, if the values of x, y, z obtained above satisfy the condition $x+y+z=a$, *i.e.*,

if $a=\frac{\sqrt{bd}}{2\sqrt{c}}+\frac{\sqrt{bc}}{2\sqrt{d}}+\frac{\sqrt{cd}}{2\sqrt{b}}$, *i.e.*, if $2a\sqrt{bcd}=bd+bc+cd$.

Otherwise the square root is not found.]

Irrational Equations

To solve equations involving surds, they are to be reduced to rational forms and then solved by various methods.

Example. Solve $\sqrt{x+6}+\sqrt{x-2}=4$. [C. U. '47]

Transposing we get $\sqrt{x+6}=4-\sqrt{x-2}$.

Squaring both sides, $x+6=16+x-2-8\sqrt{x-2}$,

or, $8\sqrt{x-2}=14-6=8$, or, $\sqrt{x-2}=1$.

Again squaring we get $x-2=1$, ∴ $x=3$.

Examples (13)

[Square roots]

Ex. 1. Find the square root of $4+2\sqrt{3}$,

Let $\sqrt{(4+2\sqrt{3})}=\sqrt{x}+\sqrt{y}$.

Squaring both sides we get $4+2\sqrt{3}=x+y+2\sqrt{xy}$.

∴ $x+y=4$...(1) and $2\sqrt{xy}=2\sqrt{3}$, or, $xy=3$...(2)

Now, $(x-y)^2=(x+y)^2-4xy=(4)^2-4\times 3=4$,

∴ $x-y=\pm 2$...(3)

Solving (1) and (3) we have $x=3,\ y=1$ or $x=1,\ y=3$.

$\therefore$ the reqd. square root $= \pm(\sqrt{3}+\sqrt{1}) = \pm(\sqrt{3}+1)$.

Ex. 2. Find the square root of $7-4\sqrt{3}$.

Here, $7-4\sqrt{3} = 7-2.2\sqrt{3} = 7-2\sqrt{4\times 3}$

$$=4+3-2\sqrt{4\times 3}=(2-\sqrt{3})^2$$

$\therefore$ the reqd. square root $= \pm(2-\sqrt{3})$.

[**N.B.** In this method first reduce the given expression to the form $a+2\sqrt{b}$. Then find two quantities whose sum is a and product is b. In the above example $4+3=7$ and $4\times 3=12$.]

Ex. 3. Find the square root of $\sqrt{48}+\sqrt{45}$.

Here $\sqrt{48}+\sqrt{45} = \sqrt{3}(\sqrt{16}+\sqrt{15}) = \sqrt{3}(4+\sqrt{15})$

$$=\sqrt{3}\times\frac{8+2\sqrt{15}}{2}$$ [To reduce $4+\sqrt{15}$ to the form $a+2\sqrt{b}$]

$$=\sqrt{3}\times\frac{5+3+2\sqrt{5.3}}{2}=\sqrt{3}\times\frac{(\sqrt{5}+\sqrt{3})^2}{2}$$

$\therefore$ the reqd. sq. root $= \pm\left\{\sqrt[4]{3}\times\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right\} = \pm\left\{\sqrt[4]{3}\left(\sqrt{\dfrac{5}{2}}+\sqrt{\dfrac{3}{2}}\right)\right\}$

Ex. 4. Find the square root of $9x+8y+12\sqrt{2xy}$.

The given exp. $=9x+8y+2.6\sqrt{2xy}$

$$=9x+8y+2\sqrt{72xy}=(\sqrt{9x}+\sqrt{8y})^2$$

$\therefore$ the reqd. sq. root $= \pm(\sqrt{9x}+\sqrt{8y}) = \pm(3\sqrt{x}+2\sqrt{2y})$.

Ex. 5. Find the square root of $9+2\sqrt{6}+4\sqrt{2}+4\sqrt{3}$.

Let $\sqrt{(9+2\sqrt{6}+4\sqrt{2}+4\sqrt{3})} = \sqrt{x}+\sqrt{y}+\sqrt{z}$.

Squaring, $9+2\sqrt{6}+4\sqrt{2}+4\sqrt{3}$

$$=x+y+z+2\sqrt{xy}+2\sqrt{xz}+2\sqrt{yz}$$

Now, this will be possible if

$x+y+z=9,\ 2\sqrt{xy}=2\sqrt{6},\ 2\sqrt{xz}=4\sqrt{2}$ and $2\sqrt{yz}=4\sqrt{3}$.

Now, $\because\ 2\sqrt{xy}=2\sqrt{6},\ \therefore\ xy=6$...(1)

$\because\ 2\sqrt{xz}=4\sqrt{2},\ \therefore\ xz=8$...(2)

and $\because\ 2\sqrt{yz}=4\sqrt{3},\ \therefore\ yz=12$...(3).

∴ From (1)×(2)×(3) we get $x^2y^2z^2=576$, ∴ $xyz=24$.

∴ $x=\frac{24}{12}=2$, $y=\frac{24}{8}=3$, $z=\frac{24}{6}=4$. These values of x, y, z satisfy the equation $x+y+z=9$.

∴ the reqd. sq. root $=\pm(\sqrt{2}+\sqrt{3}+\sqrt{4})=\pm(\sqrt{2}+\sqrt{3}+2)$.

Ex. 6. Find the square root of $17-6\sqrt{2}+4\sqrt{6}-8\sqrt{3}$.

Let $\sqrt{(17-6\sqrt{2}+4\sqrt{6}-8\sqrt{3})}=\sqrt{x}-\sqrt{y}+\sqrt{z}$.

Squaring, $17-6\sqrt{2}+4\sqrt{6}-8\sqrt{3}$

$$=x+y+z-2\sqrt{xy}+2\sqrt{xz}-2\sqrt{yz}$$

This will be possible, if $x+y+z=17$,

$-2\sqrt{xy}=-6\sqrt{2}$, $2\sqrt{xz}=4\sqrt{6}$ and $-2\sqrt{yz}=-8\sqrt{3}$.

Hence $xy=18$, $xz=24$, $yz=48$, ∴ $xyz=144$.

∴ $x=3$, $y=6$, $z=8$.

∴ The reqd. sq. root $=\pm(\sqrt{3}-\sqrt{6}+\sqrt{8})$.

Ex. 7. Find the square root of $\frac{1}{2}(3x-1)+\sqrt{2x^2+x-6}$.

The given expression $=\frac{1}{2}\{(3x-1)+2\sqrt{2x^2+x-6}\}$

$$=\tfrac{1}{2}\{(2x-3)+(x+2)+2\sqrt{(2x-3)(x+2)}\}$$

$$=\tfrac{1}{2}\{\sqrt{2x-3}+\sqrt{x+2}\}^2.$$

∴ the reqd. square root $=\pm\dfrac{1}{\sqrt{2}}\left(\sqrt{2x-3}+\sqrt{x+2}\right)$.

[**Note :** Here $2x-3$ and $x+2$ are the factors of $2x^2+x-6$ and $3x-1$ is the sum of these factors.]

Ex. 8. Find the square root of $1+a^2+\sqrt{(1+a^2+a^4)}$.

The given exp. $=1+a^2+\sqrt{(1+a+a^2)(1-a+a^2)}$

$$=\tfrac{1}{2}\{2+2a^2+2\sqrt{(1+a+a^2)(1-a+a^2)}\}$$

$$=\tfrac{1}{2}\{(1+a+a^2)+(1-a+a^2)+2\sqrt{(1+a+a^2)(1-a+a^2)}\}$$

$$=\tfrac{1}{2}\{\sqrt{(1+a+a^2)}+\sqrt{(1-a+a^2)}\}^2$$

∴ the reqd. square root

$$=\pm\frac{1}{\sqrt{2}}\{\sqrt{(1+a+a^2)}+\sqrt{(1-a+a^2)}\}.$$

4. [Rationalising factor of Binomial surds]

Examples (14)

Ex. 1. Find the rationalising factor of $\sqrt[p]{x}-\sqrt[q]{y}$.

Let $\sqrt[p]{x}$ or $x^{\frac{1}{p}}=a$ and $\sqrt[q]{y}$ or $y^{\frac{1}{q}}=b$, then the given exp. $=a-b$.

Now, if the L. C. M. of p and q be n, then a^n and b^n will both be rational, so the expression a^n-b^n will be rational.

Here a^n-b^n is divisible by $a-b$, whether n be an even or odd integer.

Now, $a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\dots\dots+b^{n-1})$

and a^n-b^n is rational.

$\therefore$ The product obtained by multiplying $a-b$ by $(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\dots\dots+b^{n-1})$ is rational.

Hence the reqd. rationalising factor

$$=a^{n-1}+a^{n-2}b+a^{n-3}b^2+\dots\dots+b^{n-1}.$$

[**N. B.** In the above example, $\because$ the L. C. M. of p and q is n, $\therefore$ if n is divided by p and q the quotients will be integers. Let the quotients be m and r respectively. Then $a^n=\left(x^{\frac{1}{p}}\right)^n$ $=x^{\frac{n}{p}}=x^m$, which is rational. $\therefore$ a^n is rational. Similarly b^n is rational.]

Ex. 2. Find the rationalising factor of $\sqrt[m]{a}+\sqrt[n]{b}$ or $a^{\frac{1}{m}}+b^{\frac{1}{n}}$.

Let $a^{\frac{1}{m}}=x$ and $b^{\frac{1}{n}}=y$, then the given expression $=x+y$.

Now, if the L. C. M. of m and n be p, then x^p and y^p are both rational and consequently x^p+y^p is rational.

(*i*) Here, if p be even, x^p-y^p is divisible by $x+y$ and then

$$x^p-y^p=(x+y)(x^{p-1}-x^{p-2}y+\dots\dots+xy^{p-2}-y^{p-1}).$$

So the product obtained by multiplying $x+y$ by the factor

$(x^{p-1}-x^{p-2}y+\ldots\ldots+xy^{p-2}-y^{p-1})$ is rational.

$\therefore$ The rationalising factor

$$=x^{p-1}-x^{p-2}y+\ldots\ldots+xy^{p-2}-y^{p-1}.$$

(*ii*) If p be odd, x^p+y^p is divisible by $x+y$ and then

$$x^p+y^p=(x+y)(x^{p-1}-x^{p-2}y+\ldots\ldots-xy^{p-2}+y^{p-1})$$

$\therefore$ The reqd. rationalising factor

$$=x^{p-1}-x^{p-2}y+\ldots\ldots-xy^{p-2}+y^{p-1}.$$

Ex. 3. Find the rationalising factor of $\sqrt{2}+\sqrt[3]{3}$.

The given exp. $=2^{\frac{1}{2}}+3^{\frac{1}{3}}$, Here the L.C.M of the denominators of the two indices is 6. Let $a=2^{\frac{1}{2}}$ and $b=3^{\frac{1}{3}}$.

Now, $a^6=\left(2^{\frac{1}{2}}\right)^6=2^3=8$, and $b^6=\left(3^{\frac{1}{3}}\right)^6=3^2=9.$

$\therefore$ a^6, b^6 and a^6-b^6 are each rational.

$\because$ $a^6-b^6=(a+b)(a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5)$,

$\therefore$ the rationalising factor of $a+b$

$$=a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5.$$

$\therefore$ The rationalising factor of $\sqrt{2}+\sqrt[3]{3}$

$$=\left(2^{\frac{1}{2}}\right)^5-\left(2^{\frac{1}{2}}\right)^4.3^{\frac{1}{3}}+\left(2^{\frac{1}{2}}\right)^3\left(3^{\frac{1}{3}}\right)^2 -\left(2^{\frac{1}{2}}\right)^2\left(3^{\frac{1}{3}}\right)^3+\left(2^{\frac{1}{2}}\right)\left(3^{\frac{1}{3}}\right)^4-\left(3^{\frac{1}{3}}\right)^5$$

$$=2^{\frac{5}{2}}-2^2 3^{\frac{1}{3}}+2^{\frac{3}{2}}.3^{\frac{2}{3}}-2.3+2^{\frac{1}{2}}.3^{\frac{4}{3}}-3^{\frac{5}{3}}$$

$$=4\sqrt{2}-4\sqrt[3]{3}+2\sqrt{2}.\sqrt[3]{9}-6+3\sqrt{2}.\sqrt[3]{3}-3\sqrt[3]{9}.$$

Ex. 4. Find the rationalising factor of $\sqrt[3]{9}-\sqrt[3]{3}+1$.

The given exp. $=9^{\frac{1}{3}}-3^{\frac{1}{3}}+1=3^{\frac{2}{3}}-3^{\frac{1}{3}}+1$

$$=\left(3^{\frac{1}{3}}\right)^2-3^{\frac{1}{3}}+1=a^2-a+1\left(\text{putting } a \text{ for } 3^{\frac{1}{3}}\right)$$

Now, multiplying a^2-a+1 by $a+1$ we get a^3+1 and it is here equal to $\left(3^{\frac{1}{3}}\right)^3+1$ or $(3+1)$ which is rational,

∴ The reqd. rationalising factor $=a+1=\sqrt[3]{3}+1$.

Ex. 5. Find the rationalising factor of $\sqrt{a}+\sqrt{b}+\sqrt{c}$.

We have $(\sqrt{a}+\sqrt{b}+\sqrt{c})\times(\sqrt{a}+\sqrt{b}-\sqrt{c})$

$=(\sqrt{a}+\sqrt{b})^2-(\sqrt{c})^2=a+b-c+2\sqrt{ab}.$

Again, multiplying $(a+b-c)+2\sqrt{ab}$ by $(a+b-c)-2\sqrt{ab}$, we have $(a+b-c)^2-(2\sqrt{ab})^2=(a+b-c)^2-4ab$ which is rational.

Now, $(a+b-c)-2\sqrt{ab}=(a+b-2\sqrt{ab})-c$

$=(\sqrt{a}-\sqrt{b})^2-(\sqrt{c})^2$

$=(\sqrt{a}-\sqrt{b}+\sqrt{c})(\sqrt{a}-\sqrt{b}-\sqrt{c}).$

∴ The reqd. rationalising factor

$=(\sqrt{a}+\sqrt{b}-\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c})(\sqrt{a}-\sqrt{b}-\sqrt{c}).$

Ex. 6. Express $\dfrac{2+\sqrt[3]{3}}{2-\sqrt[3]{3}}$ with rational denominator.

Here the denominator is $2-3^{\frac{1}{3}}$ and the indices of the terms are 1 and $\frac{1}{3}$. The L.C.M. of the denominators of the indices is 3.

∵ $a^3-b^3=(a-b)(a^2+ab+b^2)$,

∴ here the rationalising factor of $2-\sqrt[3]{3}=(2)^2+2.3^{\frac{1}{3}}+(3^{\frac{1}{3}})^2$

Now, the given fraction $=\dfrac{(2+3^{\frac{1}{3}})\{2^2+2.3^{\frac{1}{3}}+(3^{\frac{1}{3}})^2\}}{(2-3^{\frac{1}{3}})\{2^2+2.3^{\frac{1}{3}}+(3^{\frac{1}{3}})^2\}}$

$$=\frac{11+8.3^{\frac{1}{3}}+4.3^{\frac{2}{3}}}{8-3}=\frac{11+8\sqrt[3]{3}+4\sqrt[3]{9}}{5}.$$

[Irrational Equations]

For irrational equations see the chapter on Quadratic Equation.

Exercise 11

1. Find the square root of :—

(*a*) $41+6\sqrt{32}$; (*b*) $28-6\sqrt{3}$; (*c*) $\frac{1}{2}(2+\sqrt{3})$ [C. U. '24]

(*d*) $\sqrt{50}-\sqrt{48}$; (*e*) $\sqrt{175}+\sqrt{147}$; (*f*) $10\frac{3}{4}+3\sqrt{7}$.

2. Find the square root of :—

(*a*) $\frac{1}{2}(4x-3)+\sqrt{3x^2-7x+2}$, (*b*) $1+x^4+\sqrt{x^8+x^4+1}$,

(*c*) $16+2\sqrt{15}+4\sqrt{6}+4\sqrt{10}$, (*d*) $11-2\sqrt{6}+6\sqrt{2}-4\sqrt{3}$,

(*e*) $x-y+z+2\sqrt{xz-yz}$.

3. Find the rationalising factors of :—

(*a*) $\sqrt[3]{3}+1$ (*b*) $\sqrt{3}+\sqrt[3]{2}$ (*c*) $\sqrt[3]{4}+\sqrt[3]{2}+1$

(*d*) $\sqrt{x}+\sqrt{y}-\sqrt{z}$.

4. Express with a rational denominator :—

(*a*) $\dfrac{\sqrt[3]{2}+1}{\sqrt[3]{2}-1}$ (*b*) $\dfrac{3-\sqrt[3]{2}}{3+\sqrt[3]{2}}$

(*c*) $\dfrac{\sqrt{5}}{\sqrt{3}+\sqrt{5}-2\sqrt{2}}$ (*d*) $\dfrac{2}{\sqrt[3]{9}-\sqrt[3]{3}+1}$

5. Find the value of :—

$$\frac{a^2+ab+b^2}{a^2-ab+b^2}, \text{ when } a=\frac{\sqrt{2}+1}{\sqrt{2}-1} \text{ and } b=\frac{\sqrt{2}-1}{\sqrt{2}+1}.$$

6. Show that $(4+\sqrt{15})^{\frac{3}{2}}+(4-\sqrt{15})^{\frac{3}{2}}=7\sqrt{10}$.

7. Find the value of $\dfrac{3+\sqrt{5}}{2\sqrt{2}-\sqrt{(15-5\sqrt{5})}}$ correct to 2 places of decimals.

8. Find the simplest value of

$$\sqrt{[\sqrt{5}+\sqrt{\{5+8\sqrt{(9-4\sqrt{5})}\}}]}$$

9. If $(x+y)^{\frac{1}{3}}+(y+z)^{\frac{1}{3}}+(z+x)^{\frac{1}{3}}=0$, show that

$$(x+y+z)^3=9(x^3+y^3+z^3).$$

10. If $x+x\sqrt{3}=10$, find the value of x to 3 significant figures.

IMAGINARY AND COMPLEX QUANTITIES

42. The square of all positive or negative numbers (or quantities) are always positive.

So a negative quantity can have no real square root. Thus, $\sqrt{-4}$ can never represent $+2$ or -2, or $\sqrt{-x^2}$ is never equal to $+x$ or $-x$. So $\sqrt{-4}$ or $\sqrt{-x^2}$ has no real value.

But as in higher mathematics quantities such as $\sqrt{-4}$ or $\sqrt{-x}$ are often used, they must have some meaning. Just as $\sqrt{x}$ denotes a quantity whose square is x, *i. e.*, $(\sqrt{x})^2 = x$, so also $\sqrt{-x}$ may denote a quantity whose square is $-x$, *i.e.*, $(\sqrt{-x})^2 = -x$.

So these quantities, as distinct from real quantities, are called **imaginary quantities.**

$\sqrt{-1}$ is an imaginary quantity and it is denoted by the symbol **i.**

$$\therefore \quad (\sqrt{-1})^2 = \sqrt{-1} \times \sqrt{-1} = -1, \textit{ i.e.}, (i)^2 = -1.$$

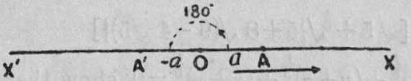
We may express any imaginary quantity as the product of a real quantity and an imaginary quantity.

$$\text{Thus, } \sqrt{-3} = \sqrt{3 \times -1} = \sqrt{3} \times \sqrt{-1} = i\sqrt{3}\,;$$

$$\sqrt{-a} = \sqrt{a \times -1} = \sqrt{a} \times \sqrt{-1} = i\sqrt{a}, \text{ etc.}$$

43. Geometrical interpretation of the symbol i

We know that any magnitude of a quantity can be expressed by lengths along a straight line.



Let OX be a straight line. We can express any magnitude (say 2 mds., 3 mds, etc.) by chosen lengths along OX. Suppose 1 centimetre denotes 1 md.

Now, the length of 2 centimetres from O towards X along OX will represent 2 mds., the length 3 cms will denote 3 mds., etc.

If we have to represent an income of Rs. 5, we may do it by 5 units of length from O along OX, but if we have to represent an expenditure of Rs. 5, we cannot do it by lengths from O to X along OX. As the expenditure of Rs. 5 is an opposite operation to income of Rs. 5, it must be represented by 5 units of length from O in the opposite direction of OX, *i.e.*, along XO produced to X′. It is, therefore, evident that when the segment of the st. line OX denoting an income of Rs. 5 turns 180° and is in the same st. line with OX in the opposite direction, it will represent an expenditure of Rs. 5 in its new position.

We know that a quantity (positive or negative) changes its sign, when multiplied by -1. So -1 denotes an operation which when performed upon any quantity changes its sign.

We have seen before that $\sqrt{-1} \times \sqrt{-1} = -1$, so it is evident that if the operation $\sqrt{-1}$ be successively performed twice in the same direction on any quantity, its sign is changed. In the figure above, let the length OA represent the quantity a. If the operation $\sqrt{-1}$ on a along OX be performed twice in succession, it will rotate the line OA by 180° (two right angles) and OA will come to the position OA′ directly opposite to OX.

So the operation $\sqrt{-1}$ means rotation through a right angle anti-clockwise and this is the geometrical interpretation of the symbol i.

So the operation denoted by $-i$ is rotation through a right angle in the clockwise direction.

44. This i is subject to all the laws of Algebra.

(*a*) **Psitive powers of i.**

$(i)^1 = i = \sqrt{-1}$; $(i)^2 = (\sqrt{-1})^2 = -1$; $(i)^3 = i^2.i = -i$,

$(i)^4 = (i^2)^2 = (-1)^2 = 1$; $(i)^5 = (i)^4.i = 1.i = i$;

$(i)^6 = (i)^4.i^2 = 1 \times -1 = -1$;

Thus, if n be any positive integer,

$$\text{then } (i)^{4n}=(i^4)^n=(1)^n=1\ ;\ (i)^{4n+1}=i^{4n}.i=i\ ;$$
$$i^{4n+2}=i^{4n}.i^2=-1,\ (i)^{4n+3}=i^{4n}.i^3=-i.$$

(*b*) **Negative powers of i**

$$(i)^{-1}=\frac{1}{i}=\frac{i}{i^2}=\frac{i}{-1}=-i\ ;\ i^{-2}=\frac{1}{i^2}=\frac{1}{-1}=-1\ ;$$

$$i^{-3}=\frac{1}{i^3}=\frac{1}{i^2.i}=\frac{1}{-i}=\frac{i}{-i^2}=i\ ;$$

$$i^{-4}=(i^{-2})^2=(-1)^2=1\ ;\ i^{-5}=\frac{1}{i^5}=\frac{1}{i^4.i}=\frac{1}{i}=\frac{i}{i^2}=-i\ ;$$

Thus generally if n be any positive integer, we have

$$i^{-4n}=\frac{1}{i^{4n}}=\frac{1}{(1)^n}=1\ ;\ i^{-(4n+1)}=\frac{1}{i^{4n+1}}=\frac{1}{i^{4n}.i}=\frac{1}{i}=-i\ ;$$

$$i^{-(4n+2)}=\frac{1}{i^{4n+2}}=-1\ ;\ \text{etc.}$$

(*c*) (i) $3i+2i=5i$; (ii) $8i-5i=3i$;

(iii) $5i\times 3i=15i^2=15\times -1=-15$; (iv) $5i\div 3i=\frac{5}{3}$;

(v) $ai\times ai=a^2i^2=a^2\times -1=-a^2$;

(vi) $ai\times bi=abi^2=-ab$.

45. Complex Quantity

If two real numbers (say a, b) are connected in the form $a+ib$ then $a+ib$ is called a **complex quantity.** This is the general form of a complex quantity. A complex quantity consists of two parts, of which **one is real** and the **other imaginary.** Thus in $a+ib$, the part a is real and the part ib is imaginary.

Here, if $b=0$, then $a+ib=a$ and consequently the complex quantity reduces to a real quantity. Again if $a=0$, then $a+ib=ib$ and so the complex quantity becomes a pure imaginary quantity.

46. Conjugate Complex Quantity

Two complex quantities, which differ only in the sign before the imaginary part, are called **conjugate complex quantities.**

Thus, $a+ib$ and $a-ib$ are two conjugate complex quantities, each being conjugate of the other.

47. Geometrical representation of a complex number.

(1) Representation of a real number geometrically by a point.

Let x be a real number, which is to be represented geometrically by a point.

Take a straight line XOX′ (fig. A). It is called the x-axis or **real axis.** A point on it represents a real number.

Let O be a fixed point on XOX′, which represents the number 0 and is called the **origin.**

Now take two points A and M on OX such that $OA=1$ and $\frac{OM}{OA}=x$. So, here the point M will represent the real number x. If x be negative ($i.e., -x$), then M will be on the straight line OX′ in the opposite direction of OX.

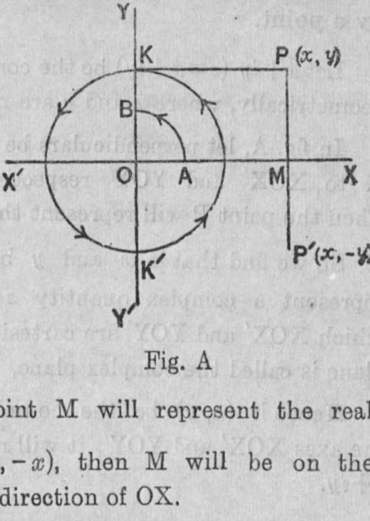


Fig. A

Hence any real number may be represented by a point on the real axis XOX′.

(2) Representation of a complex number geometrically by a point.

Let iy be a pure imaginary number in which y is real. It is to be represented geometrically by a point.

Let YOY′ be perpendicular on XOX′ through the origin O (fig. A). YOY′ (y-axis) may be called the imaginary axis. Take two points B and K on YOY′ so that OB=OA=1 and $\frac{\text{OK}}{\text{OB}}=y$. As the operation i (or $\sqrt{-1}$) on any quantity means its anti-clock-wise rotation through a right angle, so the point K will represent the imaginary number iy. If iy be negative (*i.e.*, $-iy$), it will be represented by a point K′ on OY′ opposite to OY where OK′=OK. Hence any imaginary number may be geometrically represented by a point on the imaginary axis YOY′.

(3) Representation of a complex quantity geometrically by a point.

Let $x+iy$ ($z\equiv x+iy$) be the complex quantity to be represented geometrically, where x and y are real.

In fig. A, let perpendiculars be drawn from the points M and K to XOX′ and YOY′ respectively and let them intersect at P. Then the point P will represent the complex quantity $x+iy$.

So we find that if x and y be real and $i=\sqrt{-1}$, then to represent a complex quantity $z\equiv x+iy$, we require a **plane** on which XOX′ and YOY′ are cartesian axes of co-ordinates. This plane is called the complex plane.

Hence if (x, y) be the co-ordinates of a point in the plane of the axes XOX′ and YOY′, it will represent the complex quantity $x+iy$.

[**N. B.** The complex quantity $x+iy$ is denoted by z. The complex plane may also be called the *z-plane* and the figure (A) is called the Argand Diagram. Also note that the origin O represents the complex number $0+i0$.]

(4) To represent geometrically $x-iy$, the complex number conjugate to $x+iy$, produce PM (in fig. A) to P′ so that MP′=PM.

This point P′ represents the complex number $x-iy$ (*i.e.*, z').

48. Modulus and Amplitude of x+iy (*i.e.*, z).

In Fig. A, the point P represents $x+iy$. So it may be said that the st. line OP represents the complex quantity z (*i.e.*, $x+iy$). Any straight line equal to OP and drawn parallel to it in the same sense may also represent the complex quantity $x+iy$.

Let r and θ represent OP and $\angle XOP$ respectively. Then $r^2=ON^2+PN^2=x^2+y^2$; and $\tan\theta=\frac{PN}{ON}=\frac{y}{x}$, or, $\theta=\tan^{-1}\frac{y}{x}$.

Now, the positive square root of x^2+y^2, *i.e.*, $+\sqrt{x^2+y^2}$ is called the **modulus** of $x+iy$ (*i.e.* z). It is written as *mod* z or as $|z|$, also as mod $(x+iy)$ or $|x+iy|$.

And $\theta=\tan^{-1}\frac{y}{x}$ is called the **amplitude** of z and may be written as **amp z.**

49. Geometrical representation of addition of complex quantities.

Let the points P and Q (in fig. B) represent respectively the complex quantities $a+ib$ and $c+id$. Join OP and OQ and complete the parallelogram OPRQ.

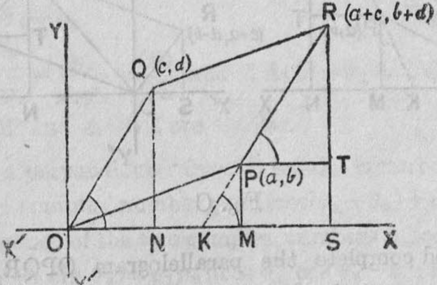


Fig. B

Then the point R will represent the sum of the two complex quantities, *i. e.*, $(a+c)+i(b+d)$.

Proof. Draw PM, QN and RS perpendiculars to OX and PT perpendicular to RS. Let RP produced meet OX at K.

$\angle RPT=\angle PKM=$ corresponding $\angle QON$.

Now, in $\triangle^s$RPT and OQN, $\angle$PTR $= \angle$QNO (being *rt.* angles), $\angle$RPT $= \angle$QON and PR $=$ OQ (opposite sides of a parallelogram),

$\therefore$ the triangles are congruent.

$\therefore$ ON $=$ PT $=$ MS, and QN $=$ RT ;

$\therefore$ OS $=$ OM $+$ MS $=$ OM $+$ ON $= a+c$,

and RS $=$ ST $+$ RT $=$ PM $+$ QN $= b+d$.

$\therefore$ R represents $(a+c)+i(b+d)$ which is the sum of $a+ib$ and $c+id$.

50. Geometrical representation of subtraction of complex quantities.

Let P and Q (in fig. C) represent respectively the complex quantities $a+ib$ and $c+id$.

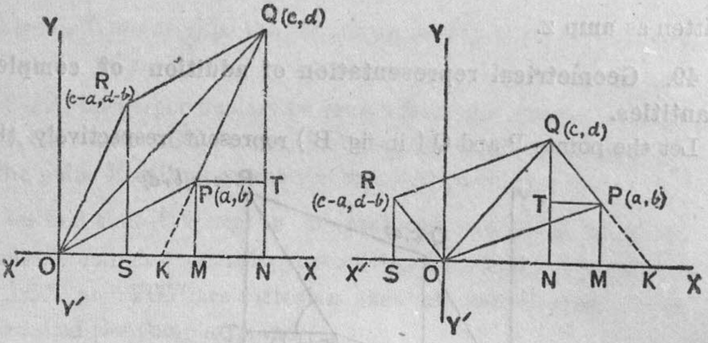


Fig. C

Join PQ and complete the parallelogram OPQR. Then the point R will represent the difference of $a+ib$ and $c+id$.

Proof. Draw PM, QN and RS perpendicular to XOX′ and PT perpendicular to QN. Let QP produced meet OX at K.

Now in $\triangle^s$PQT and ROS, $\angle$QPT $= \angle$PKM $= \angle$ROS, $\angle$QTP $= \angle$RSO, and PQ $=$ OR, $\therefore$ the $\triangle^s$ are congruent.

$\therefore$ in the first fig., OS $=$ PT $=$ NM $=$ ON $-$ OM $= c-a$,

and in the second fig., OS $=$ PT $=$ OM $-$ ON $= a-c$, but OS being negative, OS $= -(a-c) = c-a$.

Again in both the figs., RS = TQ = QN − PM = $d-b$.

$\therefore$ the point R represents $-(a-c)+i(d-b)$ or $(c-a)+i(d-b)$, *i.e.*, the difference of $a+ib$ and $c+id$.

51. Geometrical representation of multiplication of complex numbers.

In figure D, let OA = 1, OP = p_1, and $\angle$POA = θ_1;

also OQ = p_2 and $\angle$AOQ = θ_2. Let $\angle$QOR be drawn equal to θ_1 so that OR = p_1p_2.

Draw PM$\perp$OX.

Proof. OM = $p_1\cos\theta_1$ and PM = $p_1\sin\theta_1$, so P represents the complex number

$p_1(\cos\theta_1+i\sin\theta_1)$.

Similarly Q represents the number $p_2(\cos\theta_2+i\sin\theta_2)$.

Now, OA = 1, OP = p_1, OQ = p_2 and $\angle$AOR = $\theta_1+\theta_2$.

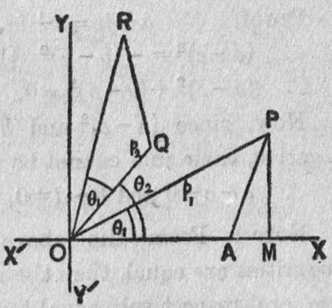


Fig. D

$$\therefore \quad \frac{OA}{OP}=\frac{1}{p_1}=\frac{p_2}{p_1p_2}=\frac{OQ}{OR}, \text{ and } \angle AOP=\theta_1=\angle QOR,$$

$\therefore$ $\triangle$AOP and $\triangle$QOR are similar.

By drawing a perpendicular from R to OX, it can be shown that R represents the complex number $p_1p_2\{\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\}$. Again, the product of the two complex numbers $p_1(\cos\theta_1+i\sin\theta_1)$ and $p_2(\cos\theta_2+i\sin\theta_2)=p_1p_2\{(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2)$

$$+i(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)\} \quad [\because i^2=-1]$$

$$=p_1p_2\{\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\}.$$

$\therefore$ It is proved that the pt. R represents the product of two complex numbers.

(8) If in fig. D, $\angle$AOR = $\theta_2-\theta_1$ and OR = $\frac{p_1}{p_2}$, it can be proved that the pt. R will represent the quotient of the two complex numbers $p_1(\cos\theta_1+i\sin\theta_1)$ and $p_2(\cos\theta_2+i\sin\theta_2)$.

52. Some properties of complex quantities.

(I) If $a+ib=0$, then $a=0$, $b=0$.

Proof. $\because$ $a+ib=0$, $\therefore$ $a=-ib$, squaring we have $a^2=-b^2$, or $a^2+b^2=0$.

Now, a^2 and b^2, being perfect squares, none is negative and so their sum cannot be 0 (zero) unless each of them is 0.

Hence $a=0$, $b=0$.

(II) If $a+ib=c+id$, then $a=c$ and $b=d$.

Proof. $\because$ $a+ib=c+id$, $\therefore$ $a-c=-i(b-d)$.

$\therefore$ $(a-c)^2=-(b-d)^2$ [by squaring, and $\because$ $i^2=-1$]

$\therefore$ $(a-c)^2+(b-d)^2=0$.

Now, since $(a-c)^2$ and $(b-d)^2$ being perfect squares, none is negative, their sum cannot be zero, unless each of them is zero.

$\therefore$ $a-c=0$ and $b-d=0$, $\therefore$ $a=c$ and $b=d$.

Note : From the above we find that if two complex quantities are equal, then the real and the imaginary parts of the one are respectively equal to the real and the imaginary parts of the other.

(III) The sum and the product of two conjugate complex quantities are both real.

Proof. Let $a+ib$ and $a-ib$ be two conjugate complex quantities.

(i) Their sum $=a+ib+a-ib=2a$, which is real.

(ii) Their product $=(a+ib)(a-ib)=a^2-i^2b^2=a^2+b^2$, which is real.

[**N. B.** The difference of two such quantities is however imaginary. Thus $(a+ib)-(a-ib)=2ib$ (imaginary)].

(IV) The sum or difference of two complex quantities is a complex quantity.

Proof. Let $a+ib$ and $c+id$ be the two complex quantities.

Now, their sum $=(a+ib)+(c+id)=(a+c)+i(b+d)$, which is a complex quantity.

Again, their difference $=(a+ib)-(c+id)=(a-c)+i(b-d)$, which is a complex quantity.

Corollary : The algebraic sum of several complex quantities is a complex quantity.

Let $a+ib$, $c+id$, $-(x+iy)$...be several complex quantities. Their sum $=(a+c-x+\ldots)+i(b+d-y+\ldots)$, which is in the form $A+iB$ and is therefore a complex quantity.

(V) The product of two or more complex quantities is a complex quantity.

(i) Let $a+ib$ and $c+id$ be the quantities.

Now, $(a+ib)(c+id)=ac+ibc+iad+i^2bd$

$=ac+i(bc+ad)-bd \quad [\because \quad i^2=-1]$

$=(ac-bd)+i(bc+ad)$, which being of the form $A+iB$ is a complex quantity.

(ii) Let $a+ib$, $c+id$, $x+iy$, be three complex quantities.

Now, $(a+ib)(c+id)(x+iy)=(ac+ibc+iad+i^2bd)(x+iy)$

$=\{(ac-bd)+i(bc+ad)\}(x+iy)$

$=(ac-bd)x+i(ac-bd)y+i(bc+ad)x+i^2(bc+ad)y$

$=(ac-bd)x+i(acy-bdy+bcx+adx)-(bc+ad)y$

$=(acx-bdx-bcy-ady)+i(acy-bdy+bcx+adx)$, which is a complex quantity being in the form of $A+iB$.

Similarly it can be proved with any number of complex quantities.

(VI) The quotient of two complex quantities is a complex quantity.

Proof. Let $a+ib$ and $c+id$ be two complex quantities.

Now, $\dfrac{a+ib}{c+id}=\dfrac{(a+ib)(c-id)}{(c+id)(c-id)}$ [multiplying the numerator and the denominator by the conjugate complex quantity of the latter.]

$$=\frac{(ac+bd)+i(bc-ad)}{c^2-i^2d^2}$$

$=\dfrac{(ac+bd)+i(bc-ad)}{c^2+d^2}=\dfrac{ac+bd}{c^2+d^2}+i\dfrac{bc-ad}{c^2+d^2}$ which is of the form $A+iB$ and so a complex quantity.

[**N. B.** If here $bc=ad$, then the quotient will be real]

(VII) Any positive integral power (square, cube, etc.) of a complex quantity is a complex quantity.

Proof. $(a+ib)^2=a^2+i^2b^2+2iab=(a^2-b^2)+2abi$, which is of the form $A+iB$ and so a complex quantity.

Again, $(a+ib)^3=a^3+3a^2ib+3ai^2b^2+i^3b^3$

$=a^3+3a^2ib-3ab^2-ib^3$ $[\because i^2=-1$ and $i^3=i^2.i=-i]$

$=(a^3-3ab^2)+i(3a^2b-b^3)$ which is a complex quantity being of the form $A+iB$.

Similarly it can be proved that any positive integral power of $(a+ib)$ is a complex quantity.

(VIII) Any root of a complex quantity is a complex quantity.

Proof. Let the nth root of $a+ib$ be x.

$\therefore \quad \sqrt[n]{a+ib}=x$, or, $(a+ib)^{\frac{1}{n}}=x$, $\therefore \quad a+ib=x^n$.

If we say x is real, then x^n is real and so $a=x^n$, for real parts of each side of the equation are equal. Hence in this case b would be zero and therefore $a+ib$ would become a real quantity, which is contrary to the hypothesis.

Hence, x is a complex quantity and so any root of a complex quantity is a complex quantity.

53. Square Root of a Complex Quantity.

Let the complex quantity be $a+ib$.

As any root of a complex quantity is a complex quantity, let us suppose $\sqrt{a+ib}=x+iy$ (where x, y are real).

Squaring both sides we have

$$a+ib=x^2+i^2y^2+2ixy=x^2-y^2+2ixy.$$

Now, equating the real parts and the imaginary parts of the two sides we have

$x^2-y^2=a$...(1) and $2ixy=ib$, or, $2xy=b$...(2).

Again, $(x^2+y^2)^2=(x^2-y^2)^2+4x^2y^2=a^2+b^2$,

$\therefore\ x^2+y^2=\sqrt{a^2+b^2}$ ......(3) [$\because$ x, y are real, $\therefore$ x^2+y^2 is positive and so the positive sq. root is taken].

Adding (3) and (1) we have

$$2x^2=\sqrt{a^2+b^2}+a, \quad \text{or,} \quad x^2=\frac{\sqrt{a^2+b^2}+a}{2}$$

$$\therefore \quad x=\pm\{\tfrac{1}{2}(\sqrt{a^2+b^2}+a)\}^{\frac{1}{2}}.$$

Again, from (3) − (1) we have

$$2y^2=\sqrt{a^2+b^2}-a, \quad \text{or,} \quad y^2=\frac{\sqrt{a^2+b^2}-a}{2}$$

$$\therefore \quad y=\pm\left\{\tfrac{1}{2}(\sqrt{a^2+b^2}-a)\right\}^{\frac{1}{2}}.$$

It is evident from (2) that xy must have the same sign as b. Hence (i) if **b is positive**, both x and y will have the same sign (both positive or both negative).

Then, the **reqd. square root**

$$=\pm[\{\tfrac{1}{2}(\sqrt{a^2+b^2}+a)\}^{\frac{1}{2}}+i\{\tfrac{1}{2}(\sqrt{a^2+b^2}-a)\}^{\frac{1}{2}}].$$

(ii) If **b is negative**, x and y will have opposite signs (one positive and the other negative).

Then the **reqd. square root**

$$=\pm[\{\tfrac{1}{2}(\sqrt{a^2+b^2}+a)\}^{\frac{1}{2}}-i\{\tfrac{1}{2}(\sqrt{a^2+b^2}-a)\}^{\frac{1}{2}}].$$

54. Modulus

The square root of each quantity has two values, equal but opposite in sign.

The positive square root of a^2+b^2 (*i.e.*, $+\sqrt{a^2+b^2}$) is called the modulus of $a+ib$. It is briefly written as mod $(a+ib)$.

Thus, mod $(a+ib)=\sqrt{(a)^2+(b)^2}=\sqrt{a^2+b^2}$,

mod $(12+5i)=+\sqrt{12^2+5^2}=13$.

mod $(x-iy)=\sqrt{(x)^2+(-y)^2}=\sqrt{x^2+y^2}$.

55. Some properties of the Modulus

1. The modulus of the product of two complex numbers is equal to the product of their moduli.

Proof. Let $a+ib$ and $c+id$ be two complex quantities. Their moduli are $\sqrt{a^2+b^2}$ and $\sqrt{c^2+d^2}$ respectively.

$\therefore$ the product of the moduli $=\sqrt{a^2+b^2}\times\sqrt{c^2+d^2}$.

Again, the product of the complex quantities

$$=(a+ib)\times(c+id)=ac+i(bc+ad)+i^2bd$$
$$=(ac-bd)+i(bc+ad)$$

$\therefore$ the modulus of the product $=\sqrt{(ac-bd)^2+(bc+ad)^2}$

$$=\sqrt{a^2c^2+b^2d^2+b^2c^2+a^2d^2}$$
$$=\sqrt{(a^2+b^2)\times(c^2+d^2)}=\sqrt{a^2+b^2}\times\sqrt{c^2+d^2}.$$

$\therefore$ mod $(a+ib)(c+id)=$ mod $(a+ib)\times$ mod $(c+id)$.

Hence the proposition is proved.

2. The modulus of the quotent of two complex quantities is the quotient of their moduli.

Proof. Let $a+ib$ and $c+id$ be two complex quantities.

Their moduli are $\sqrt{a^2+b^2}$ and $\sqrt{c^2+d^2}$ respectively.

$\therefore$ the quotient of the two moduli $=\dfrac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$.

Again, the quotient of the two complex quantities

$$=\frac{a+ib}{c+id}=\frac{(a+ib)(c-id)}{(c+id)(c-id)}$$

$$=\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}=\frac{ac+bd}{c^2+d^2}+i\,\frac{bc-ad}{c^2+d^2}$$

$\therefore$ the modulus of the quotient $=\sqrt{\left\{\left(\dfrac{ac+bd}{c^2+d^2}\right)^2+\left(\dfrac{bc-ad}{c^2+d^2}\right)^2\right\}}$

$$=\sqrt{\left\{\frac{a^2c^2+b^2d^2+b^2c^2+a^2d^2}{(c^2+d^2)^2}\right\}}=\sqrt{\left\{\frac{(a^2+b^2)(c^2+d^2)}{(c^2+d^2)^2}\right\}}$$

$$=\sqrt{\left(\frac{a^2+b^2}{c^2+d^2}\right)}=\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}.$$

$$\therefore \quad \text{mod}\left(\frac{a+ib}{c+id}\right)=\frac{\text{mod }(a+ib)}{\text{mod }(c+id)}.$$

Hence, the proposition is proved.

3. A complex number and its conjugate have the same modulus.

Let $x+iy$ and $x-iy$ be two conjugate complex quantities.

Now, mod $(x+iy)=\sqrt{(x)^2+(y)^2}=\sqrt{x^2+y^2}$,

and mod $(x-iy)=\sqrt{(x)^2+(-y)^2}=\sqrt{x^2+y^2}$.

Hence, the proposition is proved.

[**N. B.** Here, $(x+iy)(x-iy)=x^2+y^2$. Hence the modulus of any one of two conjugate complex quantities is equal to the positive square root of their product.]

56. Cube roots of unity (*i.e.*, 1). [N. B. Pre-U. '64]

Let $\sqrt[3]{1}=x$

$\therefore \quad x^3=1$, or, $x^3-1=0$, or, $(x-1)(x^2+x+1)=0$.

$\therefore \quad$ either $x-1=0\ldots(1)$, or, $x^2+x+1=0\ldots(2)$.

Now, from (1) we have $x=1$.

Again, from (2) we have $x=\dfrac{-1\pm\sqrt{1^2-4.1}}{2.1}=\dfrac{-1\pm\sqrt{-3}}{2}$.

$\therefore \quad$ the cube roots of $1=1$, $\frac{1}{2}(-1+\sqrt{-3})$ and $\frac{1}{2}(-1-\sqrt{-3})$.

[Of the three cube roots of 1, one is real and the other two are imaginary.]

57. Some properties of the cube roots of unity.

I. The sum of the three cube roots of unity is zero.

Proof. The sum $=1+\frac{1}{2}(-1+\sqrt{-3})+\frac{1}{2}(-1-\sqrt{-3})$

$$=1+\tfrac{1}{2}(-1+\sqrt{-3}-1-\sqrt{-3})$$

$$=1+\tfrac{1}{2}\times(-2)=1-1=0.$$

II. Of the two imaginary cube roots of 1, each is the square of the other.

Proof. $\frac{1}{2}(-1+\sqrt{-3})$ and $\frac{1}{2}(-1-\sqrt{-3})$, are the two imaginary cube roots of 1.

Now, $\{\frac{1}{2}(-1+\sqrt{-3})\}^2=\frac{1}{4}(1-3-2\sqrt{-3})$

$$=\tfrac{1}{4}(-2-2\sqrt{-3})=\tfrac{1}{2}(-1-\sqrt{-3}).$$

Again, $\{\frac{1}{2}(-1-\sqrt{-3})\}^2=\frac{1}{4}(1-3+2\sqrt{-3})$

$$=\tfrac{1}{4}(-2+2\sqrt{-3})=\tfrac{1}{2}(-1+\sqrt{-3}).$$

∴ Each imaginary cube root of 1 is the square of the other.

II. The product of the two imaginary cube roots of unity is unity.

Proof. $\{\frac{1}{2}(-1+\sqrt{-3})\}\{\frac{1}{2}(-1-\sqrt{-3})\}$

$$=\tfrac{1}{4}\{(-1)^2-(\sqrt{-3})^2\}=\tfrac{1}{4}(1+3)=\tfrac{1}{4}\times 4=1.$$

[**N. B.** ∵ the product of the two imaginary cube roots of 1 is 1. ∴ each is the reciprocal of the other.]

The three cube roots of unity are generally represented by $1, \omega, \omega^2$, of which the first is real but ω and ω^2 are imaginary. As each imaginary cube root is the square of the other, so if one is denoted by ω, the other is ω^2.

Thus, from the above properties, we have (by these symbols) (i) $1+\omega+\omega^2=0$ and (ii) $\omega.\omega^2=1$, or, $\omega^3=1$.

Otherwise : (i) ∵ ω is one of the roots of the equation $x^2+x+1=0$, ∴ this root will satisfy the equn., ∴ $\omega^2+\omega+1=0$.

[**Cor** : $\omega+\omega^2=-1$]

(ii) ∵ ω is also a root of the equation $x^3-1=0$,

∴ this root satisfies the equation.

∴ $\omega^3-1=0$, or, $\omega^3=1$, *i,e.*, $\omega.\omega^2=1$.

[**N. B.** Any number has three cube roots, of which one is real (this is the arithmetical cube root) and the other two imaginary. To find the cube roots of any number, first find out its arithmetical cube root and then multiply it by 1, ω and ω^2.]

58. Powers of ω

Any positive integral power of ω is equal to 1 or ω or ω^2.

We have $\omega^3=1$.

$$\therefore \quad \omega^4=\omega^3.\omega=\omega\,;\ \omega^5=\omega^3.\omega^2=\omega^2\,;\ \omega^6=(\omega^3)^2=(1)^2=1\,;$$

$$\omega^7=\omega^6.\omega=\omega\,;\ \omega^8=\omega^6.\omega^2=\omega^2\,;\ \text{etc.}$$

Generally, let n be any positive integer.

If $n=3m$ (m being an integer),

then $\omega^n=\omega^{3m}=(\omega^3)^m=(1)^m=1$;

if $n=3m+1$, then $\omega^n=\omega^{3m+1}=\omega^{3m}.\omega=1.\omega=\omega$;

if $n=3m+2$, then $\omega^n=\omega^{3m+2}=\omega^{3m}.\omega^2=1.\omega^2=\omega^2$.

Hence, $\omega^n=1$, ω or ω^2 according as n leaves the remainders 0, 1 or 2, when divided by 3.

Similarly it can be proved that any negative integral power of ω is equal to 1, ω or ω^2.

Examples (15)

Ex. 1. Find the product of $3\sqrt{-2}+4\sqrt{-3}$

and $5\sqrt{-2}-2\sqrt{-3}$.

$$\sqrt{-2}=\sqrt{2\times -1}=\sqrt{2}\times\sqrt{-1}=\sqrt{2}i, \text{ and } \sqrt{-3}=\sqrt{3}i.$$

Now, $(3\sqrt{-2}+4\sqrt{-3})(5\sqrt{-2}-2\sqrt{-3})$

$$=(3\sqrt{2}i+4\sqrt{3}i)(5\sqrt{2}i-2\sqrt{3}i)$$

$$=i(3\sqrt{2}+4\sqrt{3})\times i(5\sqrt{2}-2\sqrt{3})$$

$$=i^2(3\sqrt{2}\times5\sqrt{2}+4\sqrt{3}\times5\sqrt{2}-3\sqrt{2}\times2\sqrt{3}-4\sqrt{3}\times2\sqrt{3})$$

$$=-1(30+20\sqrt{6}-6\sqrt{6}-24)=-(6+14\sqrt{6}).$$

Ex. 2. Express $\dfrac{1}{4-\sqrt{-3}}$ with a rational denominator.

$$\frac{1}{4-\sqrt{-3}}=\frac{1}{4-i\sqrt{3}}=\frac{1\times(4+i\sqrt{3})}{(4-i\sqrt{3})(4+i\sqrt{3})}=\frac{4+i\sqrt{3}}{(4)^2-(i\sqrt{3})^2}$$

$$=\frac{4+i\sqrt{3}}{16+3}=\frac{4+\sqrt{3}i}{19}.$$

Ex. 3. Express $\dfrac{2\sqrt{-5}+3\sqrt{-3}}{2\sqrt{-5}-3\sqrt{-3}}$ with a rational denominator.

$$\frac{2\sqrt{-5}+3\sqrt{-3}}{2\sqrt{-5}-3\sqrt{-3}}=\frac{2\sqrt{5}i+3\sqrt{3}i}{2\sqrt{5}i-3\sqrt{3}i}=\frac{i(2\sqrt{5}+3\sqrt{3})}{i(2\sqrt{5}-3\sqrt{3})}$$

$$=\frac{(2\sqrt{5}+3\sqrt{3})}{2\sqrt{5}-3\sqrt{3}}=\frac{(2\sqrt{5}+3\sqrt{3})^2}{(2\sqrt{5})^2-(3\sqrt{3})^2}$$

$$=\frac{47+12\sqrt{15}}{-7}=-\frac{47+12\sqrt{15}}{7}.$$

Ex. 4. Simplify $\dfrac{x+iy}{x-iy}+\dfrac{x-iy}{x+iy}$.

The given exp. $=\dfrac{(x+iy)^2+(x-iy)^2}{(x)^2-(iy)^2}=\dfrac{2(x^2-y^2)}{x^2+y^2}$.

Ex. 5. Simplify $\dfrac{(a+i)^2}{a-i}-\dfrac{(a-i)^2}{a+i}$.

The given exp. $=\dfrac{(a+i)^3-(a-i)^3}{(a-i)(a+i)}$

$$=\frac{(a+i-a+i)\{(a+i)^2+(a+i)(a-i)+(a-i)^2\}}{(a)^2-(i)^2}$$

$$=\frac{2i(3a^2-1)}{a^2+1}=\frac{i(6a^2-2)}{a^2+1}.$$

Ex. 6. Express $\dfrac{4+3i}{3-2i}$ in the form of $A+iB$.

$$\frac{4+3i}{3-2i}=\frac{(4+3i)(3+2i)}{(3-2i)(3+2i)}=\frac{12+9i+8i+6i^2}{(3)^2-(2i)^2}=\frac{12+17i-6}{9+4}$$

$=\dfrac{6+17i}{13}=\dfrac{6}{13}+\dfrac{17}{13}i$ which is in the form of $A+iB$.

Ex. 7. Find the square root of $16-30i$. [P. U. '36]

Let $\sqrt{16-30i}=x+iy$.

Squaring both sides we have $16-30i=x^2-y^2+2ixy$

$\therefore \quad x^2-y^2=16 \ldots\ldots(1)$ and $2xy=-30 \ldots\ldots(2)$.

$$\therefore \quad (x^2+y^2)^2=(x^2-y^2)^2+(2xy)^2=(16)^2+(-30)^2$$
$$=1156=(34)^2$$

$\therefore \quad x^2+y^2=34 \ldots\ldots(3)$

Adding (3) and (1) we have $x^2=25$, $\therefore \quad x=\pm 5$.

Subtracting (1) from (3) we have $y^2=9 \quad \therefore \quad y=\pm 3$.

$\because \quad xy$ is negative here, $\therefore \quad x$ and y will have opposite signs.

$\therefore \quad$ If $x=5$, then $y=-3$, and if $x=-5$, then $y=3$.

$\therefore \quad$ the reqd. square root $=5-3i$ and $-5+3i=\pm(5-3i)$.

[**N. B.** This is the general method of finding out such square roots. In some cases the square root can easily be found by inspection.]

Ex. 8. Extract the square root of $2(21i-20)$. [P. U. '45]

$$2(21i-20)=42i-40=2.3.7i+9-49$$
$$=2.3.7i+(3)^2+(7i)^2, \quad [\because \ (7i)^2=49i^2=-49]$$
$$=(3+7i)^2. \quad \therefore \quad \text{The required sq. root}=\pm(3+7i).$$

Ex. 9. Find the square root of i.

$$\because \quad i=\tfrac{1}{2}\times 2i=\tfrac{1}{2}(1+2i-1)=\tfrac{1}{2}(1+2i+i^2)=\tfrac{1}{2}(1+i)^2,$$

$$\therefore \quad \sqrt{i}=\pm\frac{1}{\sqrt{2}}(1+i).$$

Ex. 10. Extract the square root of $\frac{1}{2}(-1+\sqrt{-3})$. [C. U. '24]

$$\tfrac{1}{2}(-1+\sqrt{-3})=\frac{-2+2\sqrt{-3}}{4}$$

$$=\frac{-3+1+2\sqrt{-3}}{4}=\frac{(\sqrt{-3}+1)^2}{4}.$$

$$\therefore \quad \text{The reqd. square root}=\pm\frac{\sqrt{-3}+1}{2}=\pm\tfrac{1}{2}(\sqrt{-3}+1).$$

Ex. 11. Find the square root of $x+i\sqrt{1-x^2}$.

$$x+i\sqrt{1-x^2}=\frac{2x+2i\sqrt{1-x^2}}{2}$$
$$=\frac{(1+x)-(1-x)+2i\sqrt{(1+x)(1-x)}}{2}$$
$$=\frac{(\sqrt{1+x}+i\sqrt{1-x})^2}{2},$$

$\therefore$ The reqd. square root $=\pm\frac{1}{\sqrt{2}}(\sqrt{1+x}+i\sqrt{1-x})$.

Ex. 12. Show that $\frac{(1+i)^2+(1-i)^2}{(1+i)^2-(1-i)^2}=0$.

Here, the numerator $=1+i^2+2i+1+i^2-2i$
$$=2+2i^2=2-2=0$$

$\therefore$ The given exp. $=\frac{0}{(1+i)^2-(1-i)^2}=0$.

Ex. 13. Show that a real value of x will satisfy the equation $\frac{1-ix}{1+ix}=a-ib$, if $a^2+b^2=1$. [B.U.E. '64 ; C. U. '33]

$$\because \quad \frac{1-ix}{1+ix}=\frac{a-ib}{1},$$
$$\therefore \quad 1-ix=a+axi-ib-i^2bx$$
$$=a+axi-ib+bx \quad [\because \; -i^2=1],$$

or, $bx+axi+ix=1-a+ib$, or, $x\{b+(a+1)i\}=(1-a)+ib$,

$\therefore \quad x=\frac{(1-a)+ib}{b+(a+1)i}$. Now, by rationalising the denominator

we have $x=\frac{\{(1-a)+ib\}\{b-(a+1)i\}}{b^2-\{(a+1)i\}^2}=\frac{2b+i(a^2+b^2-1)}{b^2+(a+1)^2}$.

Now, we find that if $a^2+b^2-1=0$, *i.e.*, if $a^2+b^2=1$, then $i(a^2+b^2-1)$ will be zero and so the value of x will be

$\frac{2b}{b^2+a^2+2a+1}$ or $\frac{2b}{2a+2}$ or $\frac{b}{a+1}$ and this value is real.

Thus it is proved that a real value of x will satisfy the given equation, if $a^2+b^2=1$.

Ex. 14. Find the modulus of $12+5i$.

Here, mod $(12+5i)=\sqrt{12^2+5^2}=\sqrt{169}=13$.

Ex. 15. Find the modulus of $(3+4i)(5+12i)$.

Here, mod $(3+4i)(5+12i)=\text{mod }(3+4i)\times\text{mod }(5+12i)$

$$=\sqrt{3^2+4^2}\times\sqrt{5^2+12^2}=\sqrt{25}\times\sqrt{169}=5\times 13=65.$$

Ex. 16. Find the modulus of $\dfrac{24+7i}{3-4i}$.

Here, mod $\dfrac{24+7i}{3-4i}=\dfrac{\text{mod }(24+7i)}{\text{mod }(3-4i)}=\dfrac{\sqrt{24^2+7^2}}{\sqrt{3^2+(-4)^2}}$

$$=\frac{\sqrt{625}}{\sqrt{25}}=\frac{25}{5}=5.$$

Ex. 17. Find the cube roots of -1.

Let, $x=\sqrt[3]{-1}$, then $x^3=-1$.

$\therefore\ x^3+1=0$, or, $(x+1)(x^2-x+1)=0$,

$$\therefore\ \left.\begin{array}{ll}\text{either,} & x+1=0\ldots\ldots(1)\\ \text{or,} & x^2-x+1=0\ldots\ldots(2)\end{array}\right\}$$

Now, from (1) we have $x+1=0$, $\therefore\ x=-1$.

Again, from (2) „ „ $x^2-x+1=0$,

$$\therefore\ x=\frac{1\pm\sqrt{1^2-4.1}}{2}=\tfrac{1}{2}(1\pm\sqrt{-3}).$$

$\therefore$ the reqd. cube roots $=-1$, $\frac{1}{2}(1+\sqrt{-3})$ and $\frac{1}{2}(1-\sqrt{-3})$.

Ex. 18. If $\dfrac{m+in}{p+iq}=x+iy$, show that $\dfrac{m-in}{p-iq}=x-iy$.

$\because\ \dfrac{m+in}{p+iq}=x+iy$, $\therefore\ \dfrac{(m+in)(p-iq)}{(p+iq)(p-iq)}=x+iy$,

or, $\dfrac{(mp+nq)+i(np-mq)}{p^2+q^2}=x+iy$,

or, $\dfrac{mp+nq}{p^2+q^2}+i\,\dfrac{np-mq}{p^2+q^2}=x+iy$,

$\therefore\ x=\dfrac{mp+nq}{p^2+q^2}$, $y=\dfrac{np-mq}{p^2+q^2}$.

Now, $\dfrac{m-in}{p-iq}=\dfrac{(m-in)(p+iq)}{(p-iq)(p+iq)}=\dfrac{(mp+nq)-i(np-mq)}{p^2+q^2}$

$$=\frac{mp+nq}{p^2+q^2}-i\frac{np-mq}{p^2+q^2}=x-iy.$$

Ex. 19. Resolve into factors $a^2+b^2+c^2-ab-bc-ca$.

[C. U. '30]

The given expression

$=a^2+b^2\omega^3+c^2\omega^3+ab(\omega+\omega^2)+bc(\omega^4+\omega^2)+ca(\omega+\omega^2)$

$[\because \ \omega^3=1,\ \omega+\omega^2=-1,\ \omega=\omega^3.\omega=\omega^4$

and $\omega^4+\omega^2=\omega+\omega^2=-1\ ]$

$=(a^2+ab\omega^2+ac\omega)+(b^2\omega^3+ab\omega+bc\omega^2)+(c^2\omega^3+bc\omega^4+ca\omega^2)$

$=a(a+b\omega^2+c\omega)+b\omega(b\omega^2+a+c\omega)+c\omega^2(c\omega+b\omega^2+a)$

$=(a+b\omega^2+c\omega)(a+b\omega+c\omega^2)$.

Ex. 20. Prove that $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)=9$.

[P. U. '39, '46]

The L. H. S. $=(1-\omega)(1-\omega^2)(1-\omega.\omega^3)(1-\omega^2.\omega^6)$

$=(1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$

$[\because \ \omega^3=1$ and $\omega^6=(1)^2=1\ ]$

$=\{(1-\omega)(1-\omega^2)\}^2=(1-\omega-\omega^2+\omega^3)^2$

$=(1-\omega+1+\omega+\omega^3)^2$

$[\because \ 1+\omega+\omega^2=0, \quad \therefore \ 1+\omega=-\omega^2\ ]$

$=(2+\omega^3)^2=(2+1)^2=9 \quad [\because \ \omega^3=1\ ]$.

Ex. 21. Show that $(\omega a+\omega^2 b)(w^2 a+\omega b)=a^2-ab+b^2$.

[C. U. '29]

$(\omega a+\omega^2 b)(\omega^2 a+\omega b)=\omega^3 a^2+\omega^2 ab+\omega^4 ab+\omega^3 b^2$

$=a^2+(\omega^2+\omega^4)ab+b^2 \quad [\because \ \omega^3=1\ ]$

$=a^2+(\omega^2+\omega^3.\omega)ab+b^2$

$=a^2+(\omega^2+\omega)ab+b^2$

$=a^2-ab+b^2 \quad [\because \ \omega^2+\omega+1=0, \quad \therefore \ \omega^2+\omega=-1\ ]$

Ex. 22. Prove that $(1+\omega-\omega^2)^3-(1-\omega+\omega^2)^3=0$.

$\because\ 1+\omega+\omega^2=0,\ \therefore\ 1+\omega=-\omega^2$ and $1+\omega^2=-\omega$.

Now, the L. H. S. $=(-\omega^2-\omega^2)^3-(-\omega-\omega)^3$

$=(-2\omega^2)^3-(-2\omega)^3=-8\omega^6+8\omega^3=-8(\omega^3)^2+8$

$=-8+8\quad[\because\ \omega^3=1]=0.$

Ex. 23. If $\omega=\frac{1}{2}(-1-\sqrt{-3})$, find the value of $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^2+\omega^{16})$. [P.U. '40]

From the given condition we have $\omega^2=\frac{1}{2}(-1+\sqrt{-3})$.

$\because\ 1+\omega+\omega^2=0,\ \therefore\ 1+\omega=-\omega^2,\ 1+\omega^2=-\omega.$

Again, $\because\ \omega^3=1,\ \therefore\ \omega^4=\omega^3.\omega=\omega,\ \omega^8=(\omega^3)^2\omega^2=\omega^2,$

and $\omega^{16}=\omega^{15}.\omega=(\omega^3)^5.\omega=(1)^5.\omega=\omega.$

Now, the given exp. $=(1-\omega+\omega^2)(1-\omega^2+\omega)(1-\omega+\omega^2)(1-\omega^2+\omega)$

$=(-2\omega)(-2\omega^2)(-2\omega)(-2\omega^2)$

$=16\omega^6=16(\omega^3)^2=16\times1=16.$

Ex. 24. If $x=a+b,\ \ y=a+b\omega,\ \ z=a+b\omega^2$, show that $x^3+y^3+z^3=3(a^3+b^3)$. [P. U. '43]

$$x^3+y^3+z^3=(a+b)^2+(a+b\omega)^3+(a+b\omega^2)^3$$

$$=a^3+b^3+3ab(a+b)+a^3+b^3\omega^3+3ab\omega(a+b\omega)$$

$$+a^3+b^3\omega^6+3ab\omega^2(a+b\omega^2)$$

$$=a^3+b^3+a^3+b^3+a^3+b^3+3ab(a+b+a\omega+b\omega^2+a\omega^2+b\omega^4)$$

$$[\because\ \omega^3=1 \text{ and } \omega^6=1]$$

$$=3(a^3+b^3)+3ab\{a(1+\omega+\omega^2)\}+b(1+\omega+\omega^2)\}$$

$$=3(a^3+b^3)\quad[\because\ 1+\omega+\omega^2=0].$$

Exercise 12

Multiply :—

1. $2\sqrt{-3}+3\sqrt{-2}$ by $5\sqrt{-3}-2\sqrt{-2}$.

2. $4\sqrt{-5}-2\sqrt{-7}$ by $5\sqrt{-5}-3\sqrt{-7}$.

3. $2\sqrt{-3}+5\sqrt{-7}$ by $4\sqrt{-6}-3\sqrt{-2}$.

Express with **a rational denominator** :—

4. $\dfrac{1}{3+\sqrt{-5}}$ **5.** $\dfrac{3+\sqrt{-5}}{2-\sqrt{-3}}$

6. $\dfrac{2-7i}{3+5i}-\dfrac{2+7i}{3-5i}$ **7.** $\dfrac{x+iy}{x-iy}-\dfrac{x-iy}{x+iy}$

8. Simplify $\dfrac{2a+ib}{a+2ib}-\dfrac{2a-ib}{a-2ib}$.

Express the following in the form of $A+iB$:—

9. $\dfrac{2+3i}{3-2i}$ **10.** $\dfrac{(2+i)^2}{3+i}$ **11.** $\dfrac{(2-i)^2}{2-3i}$ [U. U. '48]

12. $\dfrac{2\sqrt{2}-\sqrt{-3}}{\sqrt{2}-i\sqrt{3}}$ **13.** $(2+i)(3+i)$

14. $(2+i)(3+2i)(4-7i)$

Find the **square roots of** :—

15. $4+6\sqrt{-5}$. **16.** $7-30\sqrt{-2}$, [P. U. '48]

17. $-11-60\sqrt{-1}$ [P. U. '44] **18.** $7-24i$

19. $-7-24i$ [P. U. '39] **20.** $-i$

21. $2(15i-8)$. **22.** $\dfrac{-1+3\sqrt{-11}}{2}$

23. $\dfrac{23+102i}{3+2i}$. **24.** $4x^2-y^2+4xyi$

25. $3a+i\sqrt{a^4-7a^2+1}$. **26.** $a^2+\dfrac{1}{a^2}-4i\left(a-\dfrac{1}{a}\right)-6$

27. Simplify $(2+3\sqrt{-5})^{\frac{1}{2}}+(2-3\sqrt{-5})^{\frac{1}{2}}$ [P. U. '59]

Show that :—

28. $\sqrt{i}-\sqrt{-i}=\sqrt{-2}$. **29.** $\dfrac{1+i}{1-i}=i$.

30. $(2+i)^{-2}+(2-i)^{-2}=\frac{6}{25}$.

31. $\{\frac{1}{2}(1-\sqrt{-3})\}^{6n}=1$.

32. $(19+5\sqrt{-24})^{\frac{1}{2}}+(19-5i\sqrt{24})^{\frac{1}{2}}=10.$

33. Find the value of $\{\frac{1}{2}(-1+\sqrt{-3})\}^{15}$
$+\{\frac{1}{2}(-1-\sqrt{-3})\}^{15}.$

34. Find the value of $(5+12i)^{-\frac{1}{2}}+(5-12i)^{-\frac{1}{2}}.$

Find the **moduli** of the following :—

35. $4+3i$ 36. $12-5i$ 37. $(3-4i)(12+5i)$

38. $\dfrac{20+15i}{12+5i}.$ 39. $\dfrac{(2+i)(1-3i)}{(3+i)(2+3i)}.$

Resolve into factors :—

40. $x^3+y^3.$ 41. $a^3+b^3+c^3-3abc.$ 42. $x^2+1.$

43. $x^2-xy+y^2.$

44. Show that $x^2+y^2+z^2-xy-yz-zx$ is the product of two imaginary factors. [C. U. '30]

45. Find the fourth roots of unity.

Prove that :—

46. $(1+\omega-\omega^2)(1-\omega+\omega^2)=4.$

47. $(1+\omega-\omega^2)^3+(1-\omega+\omega^2)^3=-16.$

48. $(1-\omega+\omega^2)^5+(1+\omega-\omega^2)^5=32.$

49. $\dfrac{a\omega+b+1}{b\omega^2+a+\omega^2}=\omega.$

50. If $a=2+3i$ and $b=2-3i$, find the value of a^2+ab+b^2.

51. Express $(1+a^2)(1+b^2)(1+c^2)$ as the sum of two squares.

52. If $x=-2+\sqrt{-3}$, find the value of
$x^4+8x^3+24x^2+32x+17.$

53. If $a=2+i$, show that $a^4+4a^3-18a^2+4a+45=0.$

54. If $\sqrt{x-iy}=a-ib$, show that $\sqrt{x+iy}=a+ib.$

55. Prove that $\left\{\dfrac{-1+\sqrt{-3}}{2}\right\}^n+\left\{\dfrac{-1-\sqrt{-3}}{2}\right\}^n=2$, if n be a multiple of 3, and $=-1$ if n be any other integer. [P. U.]

SIMULTANEOUS EQUATIONS

(Two unknowns—One linear and the other quadratic)

59. In solving such equations, the method generally followed is to express one of the unknown quantities (x or y) in terms of the other from the given linear equation. In some cases the solution is obtained by special artifices.

Examples (16)

Ex. 16. Solve $\left.\begin{aligned} x-y&=2 \\ xy&=3 \end{aligned}\right\}\begin{aligned}&\ldots\ldots(1)\\&\ldots\ldots(2)\end{aligned}$

From (1) we have $x=2+y$.

Putting this value of x in (2) we have $(y+2)y=3$,

or, $y^2+2y-3=0$, or, $(y+3)(y-1)=0$, $\therefore\ y=1$ or -3.

Now, if $y=1$, from (1) we have $x-1=2$, $\therefore\ x=3$;

Again, if $y=-3$, from (1) we have $x+3=2$, $\therefore\ x=-1$.

Hence the solution is $\left.\begin{aligned} x&=3 \\ y&=1 \end{aligned}\right\}$ or $\left.\begin{aligned} x&=-1 \\ y&=-3 \end{aligned}\right\}$.

[**N. B.** : By the above method all such equations may be solved. The value of either x or y (according to convenience) from the linear equation is substituted in the other equation. In writing the answer the corresponding values of x and y should be grouped together.]

[**Other method**] $(x+y)^2=(x-y)^2+4xy=(2)^2+4\times3=16$,

$\therefore\ x+y=\pm4$.

Now, $\left.\begin{aligned} x+y&=\ \ 4 \\ x-y&=\ \ 2 \end{aligned}\right\}$

(Adding) $2x\ =6$, $\therefore\ x=3$. Hence $y=4-3=1$.

Again, $\left.\begin{aligned} x+y&=-4 \\ x-y&=\ \ 2 \end{aligned}\right\}$

(Adding) $2x=-2$, $\therefore\ x=-1$. Hence $y=-4+1=-3$.

$\therefore\ x=3, y=1$; or $x=-1, y=-3$.

Ex. 2. Solve $x+y=7 \ldots (1)$, $x^2+2y=17 \ldots (2)$.

From (1) we have $y=7-x \ldots (3)$.

Putting this value of y in (2) we get $x^2+2(7-x)=17$,

or, $x^2-2x-3=0$, or, $(x-3)(x+1)=0$,

$\therefore \quad x=3 \quad \text{or} \quad -1.$

Now, if $x=3$, from (3) we have $y=7-3=4$;

and if $x=-1$, „ „ „ „ $y=7+1=8$.

$$\therefore \text{ the solution is } \left.\begin{matrix} x=3 \\ y=4 \end{matrix}\right\} \quad \text{or} \quad \left.\begin{matrix} x=-1 \\ y=8 \end{matrix}\right\}.$$

Ex. 3. Solve $\left.\begin{matrix} x+y=7 \\ x^2+y^2=25 \end{matrix}\right\} \begin{matrix} \ldots\ldots(1) \\ \ldots\ldots(2) \end{matrix}$

$\because \quad x+y=7, \qquad \therefore \; x=7-y \ldots (3)$

Putting $7-y$ for x in (2) we get $(7-y)^2+y^2=25$,

or, $49-14y+2y^2=25$, or, $2y^2-14y+24=0$,

or, $y^2-7y+12=0$, or, $(y-3)(y-4)=0$, $\therefore \; y=3$ or 4.

If $y=3$, then from (3) we have $x=7-3=4$;

and if $y=4$, „ „ „ „ $x=7-4=3$.

$$\therefore \text{ The solution is } \left.\begin{matrix} x=4 \\ y=3 \end{matrix}\right\} \quad \text{or} \quad \left.\begin{matrix} x=3 \\ y=4 \end{matrix}\right\}.$$

Ex. 4. Solve $\left.\begin{matrix} x^2+xy=15 \ldots (1) \\ x - y = 1 \ldots (2) \end{matrix}\right\}$ [C. U. '15]

From (2) we have $y=x-1 \ldots (3)$

$\therefore$ From (1) $x^2+x(x-1)=15$, or, $2x^2-x-15=0$,

or, $(x-3)(2x+5)=0$, $\therefore \; x=3, -\frac{5}{2}$.

If $x=3$, from (3) we have $y=3-1=2$;

and if $x=-\frac{5}{2}$, „ „ $y=-\frac{5}{2}-1=-\frac{7}{2}$.

$$\therefore \text{ the solution is } \left.\begin{matrix} x=3 \\ y=2 \end{matrix}\right\} \quad \text{or} \quad \left.\begin{matrix} x=-2\frac{1}{2} \\ y=-3\frac{1}{2} \end{matrix}\right\}.$$

Ex. 5. Solve $\left.\begin{array}{l}\frac{1}{x}+\frac{1}{y}=\frac{1}{2}\ldots\ldots(1)\\ x+y=9\ldots\ldots(2)\end{array}\right\}$ [C. U. '36]

From (1), we have $\frac{x+y}{xy}=\frac{1}{2}$, $\therefore\ \frac{9}{xy}=\frac{1}{2}$ $[\because x+y=9]$,

$\therefore\ xy=18.$

$\therefore\ (x-y)^2=(x+y)^2-4xy=81-72=9,\quad \therefore\ x-y=\pm 3.$

Now, $x+y=9$

$x-y=3$

(Adding) $2x\ \ =12$, $\therefore\ x=6$, $\therefore$ from (2), $y=9-6=3$.

Again, $x+y=9$

$x-y=-3$

(Adding) $2x=6$, $\therefore\ x=3$, $\therefore\ y=9-3=6$.

$\therefore$ the solution is $\left.\begin{array}{l}x=6\\ y=3\end{array}\right\}$ or $\left.\begin{array}{l}x=3\\ y=6\end{array}\right\}$.

Ex. 6. Solve $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{5}{2}$, $x+y=10$. [B.U.E.'63 ; C.U. '39]

From the 1st. equation we get $\frac{x+y}{\sqrt{xy}}=\frac{5}{2}$,

or, $\frac{10}{\sqrt{xy}}=\frac{5}{2}$ $[\because\ x+y=10]$, or, $5\sqrt{xy}=20$,

or, $\sqrt{xy}=4$. $\therefore\ xy=16$,

or, $x(10-x)=16$ $[\because\ y=10-x$ from (2) $]$

or, $10x-x^2=16$, or, $x^2-10x+16=0$,

or, $(x-2)(x-8)=0$, $\therefore\ x=2$ or 8.

Now, (2), $y=8$ when $x=2$ and $y=2$ when $x=8$.

$\therefore$ The solution is $x=2$, $y=8$; or $x=8$, $y=2$.

Ex. 7. $\left.\begin{array}{l}ax^2+by^2=a+b\ldots\ldots(1)\\ x+y=1\quad\ldots\ldots(2)\end{array}\right\}$ [B.U.E. '63 ; C.U. '29]

$\because\ x+y=1$, $\therefore\ y=1-x\ldots\ldots(3)$

Now, putting $1-x$ for y in (1) we have

$$ax^2+b(1-x)^2=a+b, \quad \text{or,} \quad ax^2+b-2bx+bx^2-a-b=0,$$

$$\text{or,} \quad (a+b)x^2-2bx-a=0,$$

$$\therefore \quad x=\frac{2b\pm\sqrt{4b^2+4a(a+b)}}{2(a+b)}=\frac{b\pm\sqrt{b^2+a^2+ab}}{a+b};$$

$$\text{and } y=1-x=1-\frac{b\pm\sqrt{a^2+b^2+ab}}{a+b}=\frac{a\mp\sqrt{a^2+b^2+ab}}{a+b}.$$

Ex. 8. Solve $x+y=a+b$, $\frac{a}{x}+\frac{b}{y}=2$. [C.U. '41]

$\because \quad x+y=a+b, \quad \therefore \quad y=a+b-x$; putting this value of y in the second equation we have $\frac{a}{x}+\frac{b}{a+b-x}=2$,

$$\text{or,} \quad \frac{a}{x}-1+\frac{b}{a+b-x}-1=0,$$

$$\text{or,} \quad \frac{a-x}{x}+\frac{b-a-b+x}{a+b-x}=0, \quad \text{or,} \quad \frac{a-x}{x}-\frac{a-x}{a+b-x}=0,$$

$$\text{or,} \quad (a-x)\left(\frac{1}{x}-\frac{1}{a+b-x}\right)=0,$$

$$\therefore \quad a-x=0 \ldots (1), \quad \text{or,} \quad \frac{1}{x}-\frac{1}{a+b-x}=0 \ldots (2)$$

From (1), $x=a$, hence $y=a+b-x=a+b-a=b$.

From (2), $\frac{1}{x}-\frac{1}{a+b-x}=0$, or, $\frac{1}{x}=\frac{1}{a+b-x}$,

$$\text{or,} \quad x=a+b-x, \quad \text{or,} \quad 2x=a+b, \quad x=\tfrac{1}{2}(a+b),$$

$$\text{and} \quad \therefore \quad y=(a+b)-x=(a+b)-\tfrac{1}{2}(a+b)=\tfrac{1}{2}(a+b).$$

Hence, the solution is $\left.\begin{array}{l} x=a \\ y=b \end{array}\right\}$ or, $x=y=\frac{1}{2}(a+b)$.

Ex. 9. Solve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ldots(1)$, $\frac{x}{a}-\frac{y}{b}=1\ldots(2)$. [C.U. '25]

$$\because \left(\frac{x}{a}+\frac{y}{b}\right)^2+\left(\frac{x}{a}-\frac{y}{b}\right)^2=2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right),$$

$$\therefore \left(\frac{x}{a}+\frac{y}{b}\right)^2+(1)^2=2\times1 \text{ [From (1) and (2)]}$$

$$\text{or, } \left(\frac{x}{a}+\frac{y}{b}\right)^2=1, \quad \therefore \quad \frac{x}{a}+\frac{y}{b}=\pm1.$$

Now, $\left.\begin{array}{l}\frac{x}{a}+\frac{y}{b}=1\\ \frac{x}{a}-\frac{y}{b}=1\end{array}\right\}$ Adding, $\frac{2x}{a}=2$, $\therefore x=a$, Subtracting, $\frac{2y}{b}=0$, $\therefore y=0$.

Again, $\left.\begin{array}{l}\frac{x}{a}+\frac{y}{b}=-1\\ \frac{x}{a}-\frac{y}{b}=1\end{array}\right\}$ Adding, $\frac{2x}{a}=0$, $\therefore x=0$, Subtracting, $\frac{2y}{b}=-2$, $\therefore y=-b$.

$\therefore$ The solution is $\left.\begin{array}{l}x=a\\y=0\end{array}\right\}$ or, $\left.\begin{array}{l}x=0\\y=-b\end{array}\right\}$.

Ex. 10. Solve $\frac{a}{x}+\frac{b}{y}=2, \frac{a^2}{x^2}+\frac{b^2}{y^2}=2.$ [C.U. '10]

$$\because \left(\frac{a}{x}+\frac{b}{y}\right)^2+\left(\frac{a}{x}-\frac{b}{y}\right)^2=2\left(\frac{a^2}{x^2}+\frac{b^2}{y^2}\right),$$

$$\therefore (2)^2+\left(\frac{a}{x}-\frac{b}{y}\right)^2=2\times2 \text{ [from the given equations.]}$$

$$\text{or, } \left(\frac{a}{x}-\frac{b}{y}\right)^2=0, \therefore \frac{a}{x}-\frac{b}{y}=0.$$

Now, $\left.\begin{array}{l}\frac{a}{x}+\frac{b}{y}=2\\ \frac{a}{x}-\frac{b}{y}=0\end{array}\right\}$ Adding, $\frac{2a}{x}=2$, $\therefore x=a$ Subtracting, $\frac{2b}{y}=2$, $\therefore y=b$.

$\therefore$ The solution is $x=a$, $y=b$.

Ex. 11. Solve $\left.\begin{aligned} x+\frac{3}{y}&=2\ldots(1)\\ y+\frac{3}{x}&=-2\ldots(2)\end{aligned}\right\}$ [C.U. 1879]

From (1) we have $xy+3=2y\ldots(3)$

From (2) we have $xy+3=-2x\ldots(4)$

$\therefore$ From (3) and (4) we have $2y=-2x$, or $y=-x\ldots(5)$

Now putting $-x$ for y in (1) we have $x+\dfrac{3}{-x}=2$,

or, $x^2-3=2x$, or, $x^2-2x-3=0$,

or, $(x-3)(x+1)=0$, $\therefore$ $x=3$ or -1.

If $x=3$, then from (5) we have $y=-3$,

and if $x=-1$, from (5) we have $y=1$.

$\therefore$ the solution is $\left.\begin{aligned}x&=3\\y&=-3\end{aligned}\right\}$ or $\left.\begin{aligned}x&=-1\\y&=1\end{aligned}\right\}$.

Ex. 12. Solve $x+y=3\ldots(1)$, $2x^2-5xy+2y^2=0\ldots(2)$. [C.U. '20]

From (2) we have $2x^2-4xy-xy+2y^2=0$,

or, $(2x-y)(x-2y)=0$, $\therefore$ $2x-y=0$ or, $x-2y=0$.

Now, $\left.\begin{aligned}x+y&=3\\2x-y&=0\end{aligned}\right\}$ Adding $3x=3$, $\therefore$ $x=1$, $\therefore$ $y=3-1=2$,

Again, $\left.\begin{aligned}x+y&=3\\x-2y&=0\end{aligned}\right\}$ Subtracting, $3y=3$, $\therefore$ $y=1$, hence $x=2$.

$\therefore$ The solution is $\left.\begin{aligned}x&=1\\y&=2\end{aligned}\right\}$ or $\left.\begin{aligned}x&=2\\y&=1\end{aligned}\right\}$.

Ex. 13. Solve $xy+x+y=27\ldots(1)$, $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{2}\ldots(2)$. [C.U. '39]

From (2) we have $\dfrac{x+y}{xy}=\dfrac{1}{2}$, or, $2(x+y)=xy$.

Now from (1) we have $2(x+y)+(x+y)=27$, or, $3(x+y)=27$,

$\therefore$ $x+y=9$. $\therefore$ $xy+9=27$ [from (1)], $\therefore$ $xy=18$.

$\therefore$ $(x-y)^2=(x+y)^2-4xy=81-72=9$, $\therefore$ $x-y=\pm3$.

Now, solving $x+y=9$ and $x-y=\pm3$ we have

$\left.\begin{aligned}x&=3\\y&=6\end{aligned}\right\}$ or $\left.\begin{aligned}x&=6\\y&=3\end{aligned}\right\}$.

Ex. 14. Solve $\left.\begin{aligned}(x+a)(y+b)&=c^2\ldots(1)\\ x+y&=a+b\quad\ldots(2)\end{aligned}\right\}$ [C.U. '30]

From (2) we have $(x+a)+(y+b)=2(a+b)\ldots(3)$

[Adding $a+b$ to both sides]

Let, $x+a=p$ and $y+b=q$, so from (3) and (1) we have $p+q=2(a+b)$ and $pq=c^2$.

$\therefore\ (p-q)^2=(p+q)^2-4pq=4(a+b)^2-4c^2=4\{(a+b)^2-c^2\}$

$\therefore\ p-q=\pm 2\sqrt{(a+b)^2-c^2}$

Now, $p+q=2(a+b)$ $\ldots\ldots(4)$

$p-q=\pm 2\sqrt{(a+b)^2-c^2}$ $\ldots\ldots(5)$

Adding (4) and (5), we have $2p=2\{a+b\pm\sqrt{(a+b)^2-c^2}\}$

$\therefore\ p=a+b\pm\sqrt{(a+b)^2-c^2}$

$\therefore\ x+a=a+b\pm\sqrt{(a+b)^2-c^2}$, $\therefore\ x=b\pm\sqrt{(a+b)^2-c^2}$.

Again, subtracting (5) from (4) we have

$$2q=2\{a+b\mp\sqrt{(a+b)^2-c^2}\}$$

$\therefore\ q=a+b\mp\sqrt{(a+b)^2-c^2}$

$\therefore\ y+b=a+b\mp\sqrt{(a+b)^2-c^2}$

$\therefore\ y=a\mp\sqrt{(a+b)^2-c^2}$.

$\therefore\ x=b\pm\sqrt{(a+b)^2-c^2}$, $y=a\mp\sqrt{(a+b)^2-c^2}$.

Ex. 15. Solve $5x-2y=0\ldots\ldots(1)$, $\dfrac{3}{x^2}-\dfrac{5}{y^2}=\dfrac{11}{20}\ldots(3)$

[C. U. '50]

From (1) we have $5x=2y$, $\therefore\ x=\dfrac{2y}{5}$, $\therefore\ x^2=\dfrac{4y^2}{25}$.

From (2), we have $\dfrac{3}{\frac{4y^2}{25}}-\dfrac{5}{y^2}=\dfrac{11}{20}$, or, $\dfrac{75}{4y^2}-\dfrac{5}{y^2}=\dfrac{11}{20}$,

or, $\dfrac{55}{4y^2}=\dfrac{11}{20}$, or, $44y^2=55\times 20$, or, $y^2=\dfrac{55\times 20}{44}=25$,

$\therefore\ y=\pm 5$. Hence, $x=\dfrac{2y}{5}=\dfrac{2}{5}\times\pm 5=\pm 2$.

$\therefore\ \left.\begin{aligned}x&=2\\ y&=5\end{aligned}\right\}$ or, $\left.\begin{aligned}x&=-2\\ y&=-5\end{aligned}\right\}$.

Ex. 16. Solve $x+y=5 \ldots\ldots(1)$ $x^2+y^2=8xy \ldots\ldots(2)$.

[C. U. '17]

$\because \quad x+y=5, \quad \therefore \quad x^2+y^2+2xy=25$ (squaring),

or, $8xy+2xy=25 \quad [\because \quad x^2+y^2=8xy]$

or, $10xy=25$, or, $xy=\frac{5}{2}$.

$\therefore \quad (x-y)^2=(x+y)^2-4xy=5^2-4\times\frac{5}{2}=15,$

$\therefore \quad x-y=\pm\sqrt{15} \ldots\ldots(3)$

Now adding (1) and (3) we get $2x=5\pm\sqrt{15}$,

$\therefore \quad x=\frac{1}{2}(5\pm\sqrt{15})$

and taking (3) from (1) we have $2y=5\mp\sqrt{15}$,

$\therefore \quad y=\frac{1}{2}(5\mp\sqrt{15})$.

Ex. 17. Solve $(a-b)x+(a+b)y=a+b \ldots\ldots(1)$

$\frac{a}{x}+\frac{b}{y}=2a \ldots\ldots(2)$. [C. U. '32]

From (1) we get $(a+b)y=(a+b)-(a-b)x \ldots\ldots(3)$

From (2) we get $bx+ay=2axy$, multiplying both sides by $(a+b)$ we have $b(a+b)x+a(a+b)y=2a(a+b)xy$,

or, $b(a+b)x+a\{(a+b)-(a-b)x\}=2a\{(a+b)-(a-b)x\}x$

[putting the value of $(a+b)y$ from (3)],

or, $abx+b^2x+a(a+b)-a^2x+abx$

$=2a^2x+2abx-2a(a-b)x^2,$

or. $2a(a-b)x^2+(b^2-3a^2)x+a(a+b)=0$ [transposing]

or, $2a(a-b)x^2-(a^2-b^2)x-2a^2x+a(a+b)=0,$

or, $(a-b)x\{2ax-(a+b)\}-a\{2ax-(a+b)\}=0,$

or, $\{2ax-(a+b)\}\{(a-b)x-a\}=0,$

$\therefore$ Either, $2ax-(a+b)=0$, or, $(a-b)x-a=0$,

$\therefore \quad x=\frac{a+b}{2a}$ or $\frac{a}{a-b}$.

If $x=\frac{a+b}{2a}$, then from (3) we have

$$y=\frac{a+b}{a+b}-\frac{a-b}{a+b}x=1-\frac{a-b}{a+b}\times\frac{a+b}{2a}=1-\frac{a-b}{2a}=\frac{a+b}{2a}.$$

If $x=\dfrac{a}{a-b}$, then from (3) we have

$$y=1-\frac{a-b}{a+b}\times\frac{a}{a-b}=1-\frac{a}{a+b}=\frac{b}{a+b},$$

$\therefore\ x=y=\dfrac{a+b}{2a}$, or, $x=\dfrac{a}{a-b}$ and $y=\dfrac{b}{a+b}$.

Ex. 18. Solve $x^y=y^2$......(1)
$y^{2y}=x^4$......(2) [B. U. E. '63, C. U. '41, '45]

From (2) we have $(y^2)^y=x^4$. Now, from (1) $x^y=y^2$,

or, $(x^y)^y=(y^2)^y$, or, $(x)^{y^2}=x^4$, $\therefore\ y^2=4$, $\therefore\ y=\pm2$.

If $y=2$, from (1) we have $x^2=2^2$, $\therefore\ x=\pm2$.

If $y=-2$, from (1) we have $(x)^{-2}=(-2)^2=4$,

or, $\dfrac{1}{x^2}=4$, or, $x^2=\frac{1}{4}$, $x=\pm\frac{1}{2}$.

Hence $x=\pm2$ and $y=2$, or, $x=\pm\frac{1}{2}$ and $y=-2$.

Ex. 19. Solve $x^y=y^x$...(1), $x=2y$...(2). [C. U. '35]

$\because\ 2y=x$, $\therefore\ (2y)^y=x^y=y^x=y^{2y}=(y^2)^y$,

$\therefore\ y^2=2y$, or, $y^2-2y=0$, or, $y(y-2)=0$, $\therefore\ y=0, 2$.

If $y=0$, then from (2) we have $x=0$,

If $y=2$, ,, ,, (2) ,, ,, $x=2\times2=4$.

$\therefore$ the solution is $x=0$, $y=0$; or, $x=4$, $y=2$.

Ex. 20. Solve $8.2^{xy}=4^y$...(1), $9^x.3^{xy}=\frac{1}{27}$...(2). [C. U. '42]

From (1) we get $2^3\times2^{xy}=2^{2y}$, or, $2^{3+xy}=2^{2y}$,

$\therefore\ 3+xy=2y\cdots(9)$

From (2) we have $3^{2x}.3^{xy}=\dfrac{1}{3^3}=3^{-3}$, or, $3^{2x+xy}=3^{-3}$;

$\therefore\ 2x+xy=-3$...(4).

From (3) - (4) we get $3-2x=2y+3$, or, $-x=y$...(5)

Now, putting $-x$ for y in (3) we have $3-x^2=-2x$,

or, $x^2-2x-3=0$, or, $(x-3)(x+1)=0$, $\therefore\ x=3, -1$.

If $x=3$, from (5) we get $y=-3$.

If $x=-1$, ,, ,, ,, ,, $y=1$.

$\therefore$ the solution is $\left.\begin{matrix}x=\ \ 3\\ y=-3\end{matrix}\right\}$ or, $\left.\begin{matrix}x=-1\\ y=\ \ 1\end{matrix}\right\}$.

Ex. 21. Solve $x^3+y^3=9 \ldots (1)$, $x+y=3 \ldots (2)$. [C. U. '16]

$\because \quad x^3+y^3=9, \quad \therefore \quad (x+y)^3-3xy(x+y)=9,$

or, $(3)^3-3xy.3=9$, or, $27-9xy=9$, $\therefore \quad xy=2.$

$\therefore \quad (x-y)^2=(x+y)^2-4xy=(3)^2-4\times 2=1,$

$$\therefore \quad x-y=\pm 1 \ldots (3)$$

Now, from (2) and (3) we have,

$(a) \left.\begin{matrix} x+y=3 \\ x-y=1 \end{matrix}\right\}$ $\quad (b) \left.\begin{matrix} x+y=3 \\ x-y=-1 \end{matrix}\right\}.$

From (a), $x=2$, $y=1$, and from (b), $x=1$, $y=2$.

$\therefore$ The solution is $\left.\begin{matrix} x=2 \\ y=1 \end{matrix}\right\}$ or, $\left.\begin{matrix} x=1 \\ y=2 \end{matrix}\right\}.$

Ex. 22. Solve $2^{3x}.4^y=128 \ldots (1)$ and $9^{x+y}=27^{xy} \ldots (2)$.

From (1) we have $2^{3x}.2^{2y}=2^7$, or, $2^{3x+2y}=2^7$.

$$\therefore \quad 3x+2y=7 \ldots (3).$$

From (2) we have $(3)^{2x+2y}=3^{3xy}$, $\therefore \quad 3xy=2x+2y \ldots (4)$.

Now, from (3) we have $2y=7-3x$, $\therefore \quad y=\dfrac{7-3x}{2} \ldots (5)$.

$\therefore$ from (4), we have $3x\times\dfrac{7-3x}{2}=2x+(7-3x)=7-x$

or, $21x-9x^2=14-2x$,

or, $9x^2-23x+14=0$, or, $9x^2-14x-9x+14=0$,

or, $(x-1)(9x-14)=0$, $\therefore \quad x=1$ or $\frac{14}{9}$.

$\therefore$ from (5) we have $y=2$ or $\frac{7}{6}$.

Hence, the solution is $\left.\begin{matrix} x=1 \\ y=2 \end{matrix}\right\}$ $\left.\begin{matrix} x=\frac{14}{9} \\ y=\frac{7}{6} \end{matrix}\right\}.$

Exercise 13

Solve :—

1. $x+y=12$, $xy=35$. 2. $x+y=7$, $x^2+y^2=29$.

3. $x+y=5$, $x^2+2y=13$.

4. $x+y=\frac{5}{6}$, $\dfrac{1}{x}-\dfrac{1}{y}=1$. [C. U. '37]

5. $x^2+y^2=1$, $3x+4y=5$. [C. U. '22]

6. $x^2+xy=28$, $x-y=1$.

7. $\sqrt{x}+\sqrt{y}=2$, $x+y=3$. [M. U. 1890]

8. $x^2+y^2=a$, $x+2y=1$.

9. $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{10}{3},\ x+y=10.$ [C. U. '47]

10. $x+\frac{4}{y}=1,\ y+\frac{4}{x}=25.$ [C. U. '40, H. S. '63]

11. $\frac{1}{x^2}+\frac{1}{y^2}=\frac{10}{9},\ \frac{1}{x}+\frac{1}{y}=\frac{4}{3}.$

12. $3x-y=8,\ y^2-8x=9.$ 13. $x^2+y^2=41,\ xy=20.$

14. $2x^2+3xy+4y^2=24,\ x+3y=7.$ [C. U. '16]

15. $y^x=4,\ y^2=2^x.$ [C. U. '43]

16. $x+\frac{1}{y}=\frac{3}{2},\ y+\frac{1}{x}=3.$ [C. U. '48]

17. $3^x=9^y,\ 5^{x+y+1}=25^{xy}.$ [C. U. '46]

18. $4^x=2^y,\ (27)^{xy}=9^{y+1}.$ 19. $x^2+y^2=b,\ x+2y=1.$

20. $\frac{x^2}{y}+\frac{y^2}{x}=18,\ x+y=12.$ [C. U. '19]

21. $x^3-y^3=218,\ x-y=2.$ [C. U. '17]

22. $x^2+y^2=74,\ xy=35.$

23. $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{13}{6},\ x+y=13.$

24. $5x=2y,\ \frac{3}{x^2}-\frac{5}{y^2}=\frac{11}{20}.$ [C. U. ,50]

25. $\frac{x^2}{y}+\frac{y^2}{x}=\frac{9}{2},\ \frac{1}{x+y}=\frac{1}{3}.$ [C. U. '52]

26. $\frac{x}{2}+\frac{y}{5}=5,\ \frac{2}{x}+\frac{5}{y}=\frac{5}{6}.$ [C. U. '53]

27. $\frac{1}{x^2}+\frac{1}{y^2}=5,\ \frac{1}{x}-\frac{1}{y}=1.$ [U. P. B. '47]

28. $x+y=a+b,\ \frac{a}{x+b}+\frac{b}{y+a}=1.$ [U. P, B. '51]

29. $ax^2+by^2=a+b,\ x+y=1.$ [C. U. '29]

30. $x+y+xy=27,\ \frac{1}{x}+\frac{1}{y}=\frac{1}{2}.$ [P. U. '49]

31. $x+y-\sqrt{xy}=7,\ x^2+y^2+xy=133.$ [C. U. '51]

32. $(x-a)(y-b)=ab,\ \frac{x}{a}=\frac{y}{b}.$ [C. U. '34]

33. $(x+y)^{\frac{2}{3}}+2(x-y)^{\frac{2}{3}}=3(x^2-y^2)^{\frac{1}{3}},\ 2x-3y=4.$ [C.U. '42]

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CALCUTTA-27

CHAPTER III

TRIGONOMETRY

ANGLES OF ANY MAGNITUDE

You know that the angles in Geometry are always positive and are confined in magnitude between 0° and 360°. But in Trigonometry the angles may be of any magnitude, both positive and negative.

1. Positive angles.

Let the st. lines XOX' and YOY' intersect at right angles at O. Suppose the st. line OP revolves in the anti-clockwise direction about the point O, starting from its original position on OX. When it occupies the position OP_1, the angle POP_1 or XOP_1 traced out by it is acute and positive. If it revolves further and comes to the position OP_2, the angle XOP_2 will be a positive obtuse angle. Thus in course of its revolution, when it gradually occupies the positions OP_3 and OP_4, it forms respectively the positive reflex angle XOP_3 and the $\angle XOP_4$ which is positive and greater than 3 but less than 4 right angles. When OP makes a complete revolution, it will coincide with its original position OX and make an angle equal to four right angles (360°).

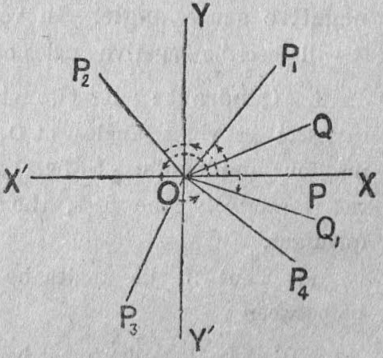

Fig. 1

Now, if after one complete revolution, it again moves in the same direction and comes to the position OQ, it evidently describes an angle which is positive and greater than four right angles.

Hence we find that OP may make any number of complete revolutions through the original position and make angles of any magnitude.

The point O is called the origin, OX the initial position and OP is called the **generating line** or **radius vector.**

The angles traced out by the radius vector revolving in the **anti-clockwise** direction are considered to be positive.

2. Negative Angles.

If in Fig. 1, the st. line OP revolves in the **clockwise** direction about O, starting from its initial position OX, and comes to the position OQ_1, the angle XOQ_1 formed by it is a **negative acute angle.** In revolving further in that direction, it will describe negative angles of any magnitude.

3. Quadrant : If two straight lines XOX' and YOY' intersect at right angles at O, the plane of the paper is divided into four **quadrants.** In fig. 1, XOY, YOX', $X'OY'$ and $Y'OX$ are respectively the first, the second, the third and the fourth quadrant.

(*i*) Thus if the angles be **positive** and if their magnitudes be between

0° and 90°, they must be in the **first quadrant,**
90° ,, 180°, ,, ,, ,, ,, ,, **second quadrant,**
180° ,, 270°, ,, ,, ,, ,, ,, **third quadrant,**
270° ,, 360°, ,, ,, ,, ,, ,, **fourth quadrant.**

(*ii*) If the angles be **negative** and if their magnitudes be between

0° and − 90°, they must lie in the **fourth quadrant,**
− 90° ,, − 180°, ,, ,, ,, ,, ,, **third quadrant,**
− 180° ,, − 270°, ,, ,, ,, ,, ,, **second quadrant,**
− 270° ,, − 360°, ,, ,, ,, ,, ,, **first quadrant.**

If an angle be 750°, then it is evident that the revolving line OP after making two complete revolutions in the anti-clockwise direction from its original position OX has traced out 30° more in the same direction, for $750° = 360° \times 2 + 30°$. Hence OP is in the first quadrant.

Again, if the angle be $-1235°$, then the revolving line OP will lie in the third quadrant. $\because \quad -1235° = -360° \times 3 - 155°$, $\therefore$ OP starting from OX has made 3 complete revolutions in the clockwise direction and traced out 155° more (*i.e.*, $-155°$) in the same negative direction. Hence it is in the third quadrant.

4. Signs of the Trigonometrical ratios.

As in graphs, the convention is to consider distances measured along OX and OY to be positive and the distances along OX′ and OY′ as negative. So the distances to the right of the y-axis are positive and those to the left are negative. Again, the distances above x-axis are positive and those below it are negative.

The revolving line OP, however, is always considered as positive in whichever quadrant it may lie.

Suppose OP is in the first quadrant as in Fig. 2 and PN⊥OX. Then in the right-angled △OPN, OP, ON and NP (the hypotenuse, base and perp.) are all positive. Hence all the trigonometrical ratios of angles in the first quadrant are positive.

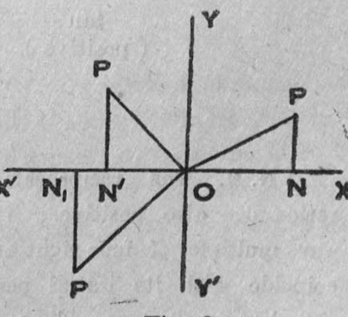

Fig. 2

If OP lies in the second quadrant and if PN′ be perpendicular to OX′, then OP and N′P are positive, but ON′ is negative. Hence, if OP be in the second quadrant, only the *sine* $\left(\frac{PN'}{OP}\right)$ of the angle and consequently its reciprocal *cosecant* are positive and the other ratios are negative.

If OP be in the third quadrant, only OP is positive and N_1P and ON_1 are negative. Hence only *tan* (and its reciprocal *cot*) is positive and all other ratios are negative.

In the fourth quadrant only PN is negative and OP, ON are positive. Hence *cos* and its reciprocal *sec* are positive in this quadrant and all other ratios are negative.

Thus, whatever may be the magnitude of the angle, positive or negative, traced out by the revolving line, the signs of trigonometrical ratios are determined according to the quadrant in which the revolving line falls.

The following figure is useful in determining the signs according to the position of OP.

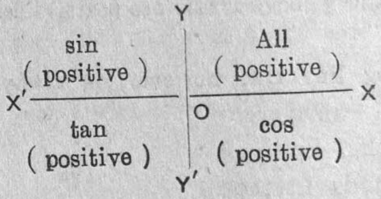

Fig. 3

[**N. B.** The reciprocals of the positive trigonometrical ratios are also positive. If an angle increases or decreases by any multiple of four right angles, its radius vector (OP) will coincide with its initial position after making one or more complete revolutions. Thus an infinite number of angles may have the same boundary line (the position of OP). These angles are called **co-terminal angles** and they are expressed by $n.360° + \theta$ (n being any integer).

Trigonometrical ratios of angles associated with a given angle θ.

5. Ratios of the angle $(-\theta)$, θ being of any magnitude. In figure 4(i) suppose OP starting from its original position OX

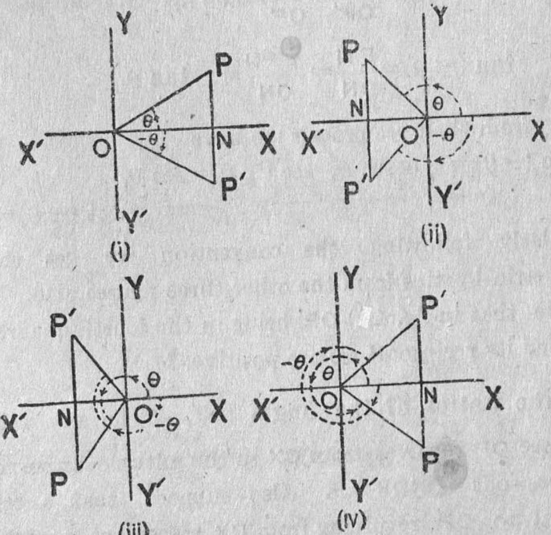

Fig. 4

revolves in the anti-clockwise (*i.e.*, positive) direction and traces out the angle XOP and let the magnitude of the angle be θ so that θ is positive. Again suppose OP revolves from OX in the clockwise (*i.e.*, negative) direction and traces out the angle XOP' equal to angle XOP. Thus $\angle XOP'$ is negative and equal to $-\theta$.

Now, draw $PN \perp XOX'$ and produce PN so that it cuts OP' at P'. Then $\angle NOP = \angle NOP'$, $\angle PNO = \angle P'NO$ (right-angles) and $OP = OP'$. $\therefore$ The right-angled Δ^s PON, $P'ON$ are congruent. Hence their corresponding sides are equal.

$\therefore\ -PN = P'N$, or, $PN = -P'N$ ($\because$ $P'N$ is negative) and $OP = OP'$ ($\because$ the radius vector OP is always positive).

Thus, $\sin(-\theta) = \dfrac{P'N}{OP'} = \dfrac{-PN}{OP} = -\sin\theta$,

$$\cos(-\theta) = \frac{ON}{OP'} = \frac{ON}{OP} = \cos\theta,$$

$$\tan(-\theta) = \frac{P'N}{ON} = \frac{-PN}{ON} = -\tan\theta.$$

Now, from their reciprocals we have
$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$, $\sec(-\theta) = \sec\theta$,
$\cot(-\theta) = -\cot\theta$.

Similarly following the convention we get the same trigonometrical ratios from the other three figures also.

[Note that in fig. 4(i) OP' being in the fourth quadrant, only the *cos* and its reciprocal *sec* are positive.]

6. The Ratios of the angle $(90^\circ - \theta)$ for any value of θ.

Suppose OP revolves from OX in the anti-clockwise direction and traces out $\angle XOP = \theta$. Also suppose that another line OP' equal to OP revolving from OX traces out a right angle, *i.e.*, coincides with OY and then revolves back in the clock-wise direction tracing out $\angle YOP' = \theta$. Then we have $\angle XOP' = 90^\circ - \theta$.

Draw PN and $P'N'$ perpendicular to XOX' and $P'M$ perp. to YOY'. Then in each figure in Fig. (5), $\angle XOP$ and $\angle YOP'$ are equal in magnitude.

$\therefore$ In each fig. $\angle PON = \angle MOP' = \angle OP'N'$ ($\because$ $OM \parallel P'N'$), and the right-angled $\triangle^s OPN$, $OP'N'$ are congruent.

$\therefore$ $ON = P'N'$ (in magnitude), $PN = ON'$ (in magnitude) and $OP = OP'$.

Again, in each figure ON and $P'N'$ have the same sign (positive or negative) and PN and ON' have the same sign.

Thus, $$\sin(90^\circ-\theta)=\sin\angle XOP'=\frac{P'N'}{OP'}=\frac{ON}{OP}=\cos\theta,$$

$$\cos(90^\circ-\theta)=\cos\angle XOP'=\frac{ON'}{OP'}=\frac{PN}{OP}=\sin\theta,$$

$$\tan(90^\circ-\theta)=\tan\angle XOP'=\frac{P'N'}{ON'}=\frac{ON}{PN}=\cot\theta.$$

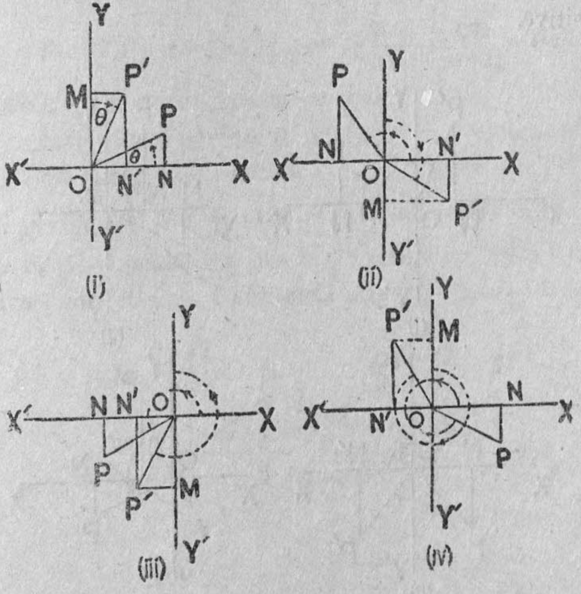

Fig. 5

Since their reciprocals are equal,

$\therefore$ $\operatorname{cosec}(90^\circ-\theta)=\sec\theta$, $\sec(90^\circ-\theta)=\operatorname{cosec}\theta$,

$\cot(90^\circ-\theta)=\tan\theta$.

[**N. B.** Here the angles $(90^\circ-\theta)$ and θ are complementary angles, their sum being one right angle. Hence we find that if two angles be complementary, the sine of one is the cosine of the other and *viceversa*. Similarly the tangent of one is the cotangent of the other and *viceversa* and the secant of one is the cosecant of the other and *viceversa*.]

7. Ratios of the angle $(90°+\theta)$ for all values of θ.

Suppose the radius vector OP revolving in the positive direction from OX traces out the angle $XOP=\theta$ [Fig. 6(i)] and then revolves further from OP in the same direction and comes to the position OP′ tracing out a right angle POP′. Then $\angle XOP'=90°+\theta$. Here OP and OP′ are always equal and positive.

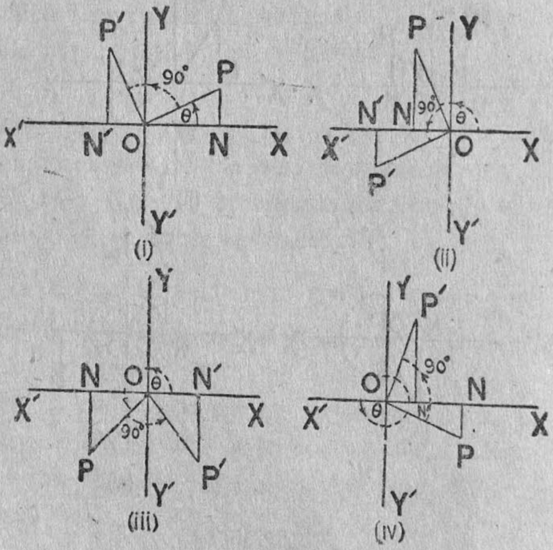

Fig. 6

From P and P′ draw PN and P′N′ perpendiculars to XOX′. Now, in all the above figures POP′ being a right angle, the sum of $\angle PON$ and $\angle P'ON'$ is a right angle.

Hence $\angle PON=90°-\angle P'ON'=\angle OP'N'$.

∴ The Δ^s PON, P′ON′ are congruent.

∴ The corresponding sides ON and P′N′ and the sides PN and ON′ are numerically equal.

Again, in each figure, ON and P′N′ have the same sign (positive or negative), but PN and ON′ are of opposite signs, *i.e.*, $P'N' = +ON$ and $ON' = -PN$.

Hence, $\textbf{sin}\ (90^\circ+\theta) = \sin \angle XOP' = \dfrac{P'N'}{OP'} = \dfrac{ON}{OP} = \cos\theta$,

$$\cos(90^\circ+\theta) = \cos\angle XOP' = \frac{ON'}{OP'} = \frac{-PN}{OP} = -\sin\theta,$$

$$\tan(90^\circ+\theta) = \tan\angle XOP' = \frac{P'N'}{ON'} = \frac{ON}{-PN} = -\cot\theta$$

Again, ∵ their reciprocals are equal,

∴ $\operatorname{cosec}(90^\circ+\theta) = \sec\theta$, $\sec(90^\circ+\theta) = -\operatorname{cosec}\theta$,
$\cot(90^\circ+\theta) = -\tan\theta$.

9. Ratios of angles (180° − θ) for all values of θ.

In fig. 7, suppose the line OP revolving from OX in the positive direction (anti-clockwise) first traces out

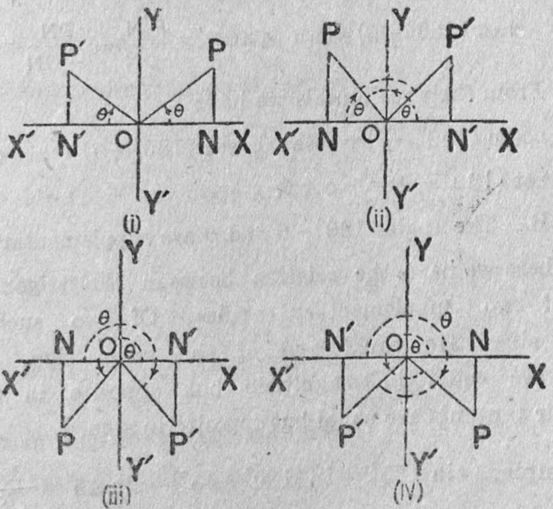

Fig. 7

$\angle XOP = \theta$ and then revolving further in the same direction coincides with OX′ (*i.e.*, tarces out two right angles or 180°).

It then revolves back in the clokwise direction and tracing out $\angle X'OP'$ equal to θ, comes to the position OP'. Thus $\angle XOP$ and $\angle X'OP'$ are equal in magnitude but opposite in sign, $\angle XOP$ being positive and $\angle X'OP'$ negative. Now we have $\angle XOP' = 180° - \theta$. Make OP' equal to OP and from P and P' draw PN and $P'N'$ perpendiculars to XOX'.

Now, in each figure $\angle PON = \angle P'ON'$ and $OP = OP'$,

∴ the right-angled $\triangle^s PON$, $P'ON'$ are congruent.

∴ In each figure ON and ON' are equal but opposite in sign, while PN and $P'N'$ are equal and of the same sign.

∴ $ON' = -ON$ and $P'N' = +PN$.

$$\text{Hence, } \mathbf{sin}\ (180° - \theta) = \sin\ \angle XOP' = \frac{P'N'}{OP'} = \frac{PN}{OP} = \mathbf{sin}\ \theta,$$

$$\mathbf{cos}\ (\mathbf{180°} - \theta) = \cos\ \angle XOP' = \frac{ON'}{OP'} = \frac{-ON}{OP} = -\mathbf{cos}\ \theta,$$

$$\mathbf{tan}\ (\mathbf{180°} - \theta) = \tan\ \angle XOP' = \frac{P'N'}{ON'} = \frac{PN}{-ON} = -\mathbf{tan}\ \theta.$$

∴ From their reciprocals we have

$\operatorname{cosec} (180° - \theta) = \operatorname{cosec} \theta$, $\sec (180° - \theta) = -\sec \theta$,

$\cot (180° - \theta) = -\cot \theta$.

[**N. B.** The angles $180° - \theta$ and θ are supplementary.

So, here we have the relation between the trigonometrical ratios of two supplementary angles. Of two such angles (i) their sines are equal and of the same sign, (ii) their cosines are equal in magnitude but opposite in sign and (iii) their tangents are equal but opposite in sign.]

Example. $\sin 135° = \sin (180° - 45°) = \sin 45° = \frac{1}{\sqrt{2}}$.

$\cos 120° = \cos (180° - 60°) = -\cos 60° = -\frac{1}{2}$.

$\tan \frac{5\pi}{6} = \tan \left(\pi - \frac{\pi}{6}\right) = -\tan \frac{\pi}{6} = -\tan 30° = -\frac{1}{\sqrt{3}}$.

9. Ratios of angle $(180^\circ+\theta)$ for all values of θ.

Suppose OP revolving from OX in the positive direction traces out angle θ and comes to the position OP. It then revolves further in the same direction and tracing out an angle of 180° comes to the position OP′ (where OP′ is in the same st. line with OP and opposite to it in direction).

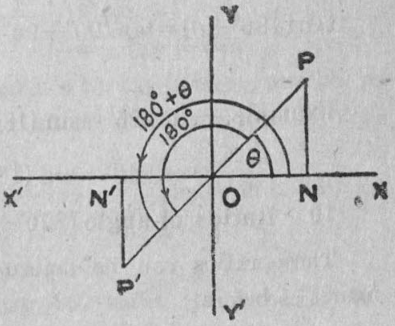

Fig. 8

Then $\angle XOP'=180^\circ+\theta$. Make $OP'=OP$, and draw PN and P′N′ perpendiculars to XOX′.

Now, $\because$ $OP=OP'$ and $\angle PON=$ vert. opp. $\angle P'ON'$,

$\therefore$ the right-angled $\triangle^s$PON, P′ON′ are congruent.

$\therefore$ their corresponding sides are equal.

$\therefore$ $P'N'=-PN$, $ON'=-ON$ and $OP'=OP$.

Hence, **sin $(180^\circ+\theta)$** $=\sin XOP'=\dfrac{P'N'}{OP'}=\dfrac{-PN}{OP}=-\sin\theta$,

cos $(180^\circ+\theta)$ $=\cos XOP'=\dfrac{ON'}{OP'}=\dfrac{-ON}{OP}=-\cos\theta$,

tan $(180^\circ+\theta)$ $=\tan XOP'=\dfrac{P'N'}{ON'}=\dfrac{-PN}{-ON}=\dfrac{PN}{ON}=\tan\theta$.

$\therefore$ from their reciprocals we have

$\operatorname{cosec}(180^\circ+\theta)=-\operatorname{cosec}\theta$, $\sec(180^\circ+\theta)=-\sec\theta$,

$\cot(180^\circ+\theta)=\cot\theta$.

[In the fig., OP is in the first quadrant. If it be in any other quadrant the figure can be easily drawn.]

[**N. B. Another Proof :** $\sin(\mathbf{180^\circ}+\theta)=\sin(90^\circ+\overline{\mathbf{90^\circ}+\theta})$
$=\cos(\mathbf{90^\circ}+\theta)=-\sin\theta$;
$\cos(180^\circ+\theta)=\cos(90^\circ+\overline{\mathbf{90^\circ}+\theta})=-\sin(90^\circ+\theta)=-\cos\theta$;
$\tan(180^\circ+\theta)=\tan(90^\circ+\overline{90^\circ+\theta})=-\cot(90^\circ+\theta)$
$=\tan\theta$; etc.]

Example. $\sin 225^\circ=\sin(180^\circ+45^\circ)=-\sin 45^\circ=-\dfrac{1}{\sqrt{2}}$.
$\cos 240^\circ=\cos(180^\circ+60^\circ)=-\cos 60^\circ=-\frac{1}{2}$.

10. Ratios of angle (270°+θ).

These ratios can be determined geometrically by drawing figures as before.

An alternative method is shown below :

$\sin(270^\circ+\theta)=\sin(180^\circ+\overline{90^\circ+\theta}=-\sin(90^\circ+\theta)=-\cos\theta$;

$\cos(270^\circ+\theta)=\cos(180^\circ+\overline{90^\circ+\theta})=-\cos(90^\circ+\theta)$
$=-(-\sin\theta)=\mathbf{\sin\theta}$;

$$\tan(270^\circ+\theta)=\frac{\sin(270^\circ+\theta)}{\cos(270^\circ+\theta)}=\frac{-\cos\theta}{\sin\theta}=-\mathbf{\cot\theta}.$$

From their reciprocals we have
$\operatorname{cosec}(270^\circ+\theta)=-\sec\theta$, $\sec(270^\circ+\theta)=\operatorname{cosec}\theta$,
$\cot(270^\circ+\theta)=-\tan\theta$.

11. Ratios of angles (360°±θ) and (n.360°±θ).

We have seen that in whatever position the revolving line OP may be after describing the angle θ, it will again be in the same position after it has made one more complete revolution or made a number of complete revolutions more in the positive direction. Hence it is evident that angles that differ by 360° or complete multiples of 360° have the same trigonometrical ratios in magnitude and in sign.

So trigonometrical ratios of $(360^\circ-\theta)$ and of $-\theta$ must be identical.

Hence, $\mathbf{\sin (360^\circ+\theta)=\sin \theta}$, $\mathbf{\cos (360^\circ+\theta)=\cos \theta}$, $\mathbf{\tan (360^\circ+\theta)=\tan \theta}$.

Again, $\mathbf{\sin (360^\circ-\theta)=\sin (-\theta)=-\sin \theta}$

$\mathbf{\cos (360^\circ-\theta)=\cos (-\theta)=\cos \theta}$

$\mathbf{\tan (360^\circ-\theta)=\tan (-\theta)=-\tan \theta}$, etc.

It may be briefly stated that if n be any integer, positive or negative, the trigonometrical ratios of $n.360^\circ\pm\theta$ (or $2n\pi\pm\theta$) must be the same as those of $\pm\theta$.

Thus it is evident that addition or subtraction of 360° or any multiple of 360° to an angle does not change its trigonometrical ratios.

12. The following rule may be made to remember the above results :

(a) **If θ be associated with any even multiple of 90° by the + or − sign,** its trigonometrical **ratios remain unaltered** (*i.e.*, sine remains sine, cosine remains cosine, etc.). To determine the sign of the ratios first find out the quadrant in which the associated angle lies, taking θ to be acute and then apply the rule "**All, sin, tan, cos.**"

(b) **If θ be associated with any odd multiple of 90° by the + or − sign, the ratios are altered** (*i.e.* sine becomes cosine, cosine becomes sine, tangent becomes cotangent, etc). The sign of each ratio is determined as in (a).

Examples (1)

Ex. 1. Find the values of (i) $\sin 480^\circ$, (ii) $\cos 405^\circ$, (iii) $\tan(-1485^\circ)$ and (iv) $\cot 1410^\circ$.

(i) Here $480^\circ=5\times 90^\circ+30^\circ$.

$\because$ the multiple of 90° is odd (5), the sine will change into cosine. Again, the angle 480° being in the second quadrant, its sine will be positive.

$$\therefore \quad \sin 480^\circ=\sin(5\times 90^\circ+30^\circ)=\cos\ 30^\circ=\frac{\sqrt{3}}{2}.$$

(ii) $405^\circ = 5\times 90^\circ - 45^\circ$, here the multiple of 90° being odd, the cosine will become sine and the angle of 405° will be in the first quadrant and its cosine will be positive.

$$\therefore \quad \cos 405^\circ = \cos(5\times 90^\circ - 45^\circ) = \sin 45^\circ = \frac{1}{\sqrt{2}}.$$

(iii) $\because \tan(-\theta) = -\tan\theta$,

$$\therefore \quad \tan(-1485^\circ) = -\tan 1485^\circ.$$

Now, $1485^\circ = 16.90^\circ + 45^\circ$. Here 16, the multiple of 90°, being even the ratio will not change. Again the angle being in the first quadrant its tan will be positive.

$$\therefore \quad \tan(-1485^\circ) = -\tan 1485^\circ = -(\tan 16.90^\circ + 45^\circ)$$
$$= -\tan 45^\circ = -1.$$

(iv) $1410^\circ = 16.90^\circ - 30^\circ$, here the multiple of 90° is even, so the ratio will not change and 1410° being in the fourth quadrant its cot will be negative.

$$\therefore \quad \cot 1410^\circ = \cot(16.90^\circ - 30^\circ) = -\cot 30^\circ = -\sqrt{3}.$$

Ex. 2. Find the sine, cosine and tangent of the following angles : (a) 180°, (b) 270°, (c) 2π.

(a) $\because \sin(180^\circ - \theta) = \sin\theta$,

$\therefore \sin(180^\circ - 0^\circ) = \sin 0^\circ$ [taking $\theta = 0^\circ$]

$\therefore \sin 180^\circ = \sin 0^\circ = 0.$

Again, $\because \cos(180^\circ - \theta) = -\cos\theta$,

$\therefore \cos(180^\circ - 0^\circ) = -\cos 0^\circ$ [taking $\theta = 0^\circ$]

$\therefore \cos 180^\circ = -\cos 0^\circ = -1.$

Hence, $\tan 180^\circ = \dfrac{\sin 180^\circ}{\cos 180^\circ} = \dfrac{0}{-1} = 0.$

(b) $\because \sin(270^\circ + \theta) = -\cos\theta$,

$\therefore \sin 270^\circ = -\cos 0^\circ$ [taking $\theta = 0^\circ$]

$= -1.$

Similarly, $\cos 270^\circ = \cos(270^\circ + 0^\circ) = \sin 0^\circ = 0$,

and $\tan 270^\circ = \dfrac{\sin 270^\circ}{\cos 270^\circ} = \dfrac{-1}{0} = -\infty.$

(c) $2\pi = 2 \times 180° = 360°$.

$\because \quad \sin(360° + \theta) = \sin\theta$,

$\therefore \quad \sin(360° + 0°) = \sin 0°$ [taking $\theta = 0°$]

$\therefore \quad \sin 2\pi = \sin 360° = \sin 0° = 0$.

Similarly, $\cos 360° = \cos 0° = 1$,

$$\therefore \quad \tan 360° = \frac{\sin 0°}{\cos 0°} = \frac{0}{1} = 0.$$

Ex. 3. Find the smallest positive coterminal angle and the value of $\cot \frac{17\pi}{4}$.

$$\frac{17\pi}{4} = 4\pi + \frac{\pi}{4} = 2 \times 360° + 45° \; [\because \; 2\pi = 360°]$$

$\therefore$ The coterminal angle $= 45°$ or $\frac{\pi}{4}$.

Again, $\cot \frac{17\pi}{4} = \cot(2 \times 360° + 45°) = \cot 45° = 1$.

Ex. 4. Find the value of

$3 \cos 270° \sec 180° + 2 \operatorname{cosec} 90° - \cos 360°$.

$$\text{The given exp.} = 3 \times \sin 0° \times \frac{1}{-\cos 0°} + 2 \times \operatorname{cosec} 90° - \cos 0°$$

$$= 3 \times 0 \times \frac{1}{-1} + 2 \times 1 - 1 = 0 + 2 - 1 = 1.$$

Ex. 5. Find all the angles numerically less than $360°$ which satisfy the equation $\sin A = -\frac{1}{2}$.

$$\because \quad \tfrac{1}{2} = \sin 30°, \quad \therefore \quad -\tfrac{1}{2} = -\sin 30°.$$

Hence all the angles, less than $360°$, whose sine $= -\sin 30°$, will satisfy the given equation.

Now $\sin(-30°) = -\sin 30° \cdots\cdots (1)$.

Again, $\sin(180° - 30°) = \sin 30°$,

or, $\sin 150° = \sin 30°$,

$\therefore \quad \sin(-150°) = -\sin 150° = -\sin 30° \cdots\cdots (2)$

Again, $\sin(180^\circ+30^\circ)=-\sin 30^\circ$

or, $\sin 210^\circ=-\sin 30^\circ\cdots\cdots(3)$.

Also, $\sin(360^\circ-30^\circ)=-\sin 30^\circ$,

or, $\sin 330^\circ=-\sin 30^\circ\cdots\cdots(4)$

$\therefore\ A=-30^\circ,\ -150^\circ,\ 210^\circ$ and 330°.

Ex. 6. Prove that

$$\sin 420^\circ\cos 390^\circ+\cos(-300^\circ)\sin(-330^\circ)=1$$

$$\begin{aligned}\text{L. H. S.}&=\sin(360^\circ+60^\circ)\cos(360^\circ+30^\circ)\\&\quad+\cos(-360^\circ+60^\circ)\sin(-360^\circ+30^\circ)\\&=\sin 60^\circ\cos 30^\circ+\cos 60^\circ\sin 30^\circ\\&=\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}+\frac{1}{2}\times\frac{1}{2}=\frac{3}{4}+\frac{1}{4}=1.\end{aligned}$$

Ex. 7. Express the following in terms of the ratios of a positive angle less than 45° :

(i) $\tan(-1385^\circ)$ and (ii) $\cos 294^\circ$.

Here, (i) $\tan(-1385^\circ)=\tan(-4\times360^\circ+55^\circ)=\tan 55^\circ$

$$=\tan(90^\circ-35^\circ)=\cot 35^\circ.$$

(ii) $\cos 294^\circ=\cos(3\times90^\circ+24^\circ)=\sin 24^\circ$.

Ex. 8. Find the simplest value of $\dfrac{\cos 255^\circ+\tan 285^\circ}{\cot 165^\circ-\sin 375^\circ}$.

$$\frac{\cos 255^\circ+\tan 285^\circ}{\cot 165^\circ-\sin 375^\circ}=\frac{\cos(270^\circ-15^\circ)+\tan(270^\circ+15^\circ)}{\cot(180^\circ-15^\circ)-\sin(360^\circ+15^\circ)}$$

$$=\frac{-\sin 15^\circ-\cot 15^\circ}{-\cot 15^\circ-\sin 15^\circ}=1.$$

Ex. 9. If $\tan\theta=-\frac{12}{5}$, find $\sin\theta$ and $\cos\theta$.

Here $\tan\theta$ being negative the boundary line of θ must be either in the second or in the 4th quadrant. [Draw the fig.] In the second quadrant the base ON is negative and the perp. PN is positive. $\therefore\ ON=-5,\ PN=12$.

$$\therefore\ OP=\sqrt{PN^2+ON^2}=\sqrt{12^2+(-5)^2}=\sqrt{169}=13.$$

Hence, $\sin\theta=\dfrac{PN}{OP}=\dfrac{12}{13}$ and $\cos\theta=\dfrac{ON}{OP}=-\dfrac{5}{13}$.

Again, in the 4th quadrant the base ON′ is positive and the perpendicular P′N′ is negative.

$\therefore$ ON′ = 5 and P′N′ = − 12 and perp. OP′ = OP = 13.

$$\therefore \quad \sin\theta = \frac{P'N'}{OP} = -\frac{12}{13} \text{ and } \cos\theta = \frac{ON'}{OP'} = \frac{5}{13}.$$

$$\text{Hence, } \sin\theta = \pm\frac{12}{13} \text{ and } \cos\theta = \pm\frac{5}{13}.$$

Ex. 10. Prove that $\cos^2\frac{\pi}{4} + \sin^2\frac{3\pi}{4} + \sin^2\frac{5\pi}{4}$

$$+\sin^2\frac{7\pi}{4} = 2.$$

$$\text{L. H. S.} = \cos^2\frac{\pi}{4} + \sin^2\left(\pi - \frac{\pi}{4}\right) + \sin^2\left(\pi + \frac{\pi}{4}\right) + \sin^2\left(2\pi - \frac{\pi}{4}\right)$$

$$= \left(\cos\frac{\pi}{4}\right)^2 + \left(\sin\frac{\pi}{4}\right)^2 + \left(-\sin\frac{\pi}{4}\right)^2 + \left(-\sin\frac{\pi}{4}\right)^2$$

$$= \left(\cos\frac{\pi}{4}\right)^2 + 3\sin^2\frac{\pi}{4} = \left(\frac{1}{\sqrt{2}}\right)^2 + 3\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{3}{2} = 2.$$

Ex. 11. If $\tan\theta = \frac{5}{12}$ and $\cos\theta$ is negative, find the value of $\dfrac{\sin\theta + \cos(-\theta)}{\sec(-\theta) + \tan\theta}$.

$\because \quad \tan\theta = \frac{5}{12}, \ \therefore \ \sec^2\theta = 1 + \tan^2\theta = 1 + \frac{25}{144} = \frac{169}{144},$

$\therefore \quad \sec\theta = \pm\frac{13}{12}.$ Here $\because$ $\cos\theta$ is negative (Hyp.),

$\therefore \quad \sec\theta = -\frac{13}{12}. \ \therefore \ \cos\theta = -\frac{12}{13}.$

Hence, $\sin\theta = \tan\theta \times \cos\theta = \frac{5}{12} \times -\frac{12}{13} = -\frac{5}{13}.$

$$\text{Now, the given exp.} = \frac{\sin\theta + \cos\theta}{\sec\theta + \tan\theta} = \frac{-\frac{5}{13} - \frac{12}{13}}{-\frac{13}{12} + \frac{5}{12}} = \frac{51}{26}.$$

Ex. 12. Solve $\cos\theta + \sqrt{3}\sin\theta = 2$ for θ, giving all the possible values, when $0° < \theta < 360°$. [C. U. 1936]

$$\begin{aligned} & \cos\theta + \sqrt{3}\sin\theta = 2, \\ \text{or,} \quad & \cos\theta - 2 = -\sqrt{3}\sin\theta, \\ \text{or,} \quad & \cos^2\theta - 4\cos\theta + 4 = 3\sin^2\theta = 3(1 - \cos^2\theta). \end{aligned}$$

$$\text{or,}\quad 4\cos^2\theta-4\cos\theta+1=0,$$

$$\text{or,}\quad (2\cos\theta-1)^2=0,\ \text{or,}\ 2\cos\theta-1=0,$$

$$\text{or,}\quad 2\cos\theta=1,\ \therefore\ \cos\theta=\tfrac{1}{2}=\cos 60^\circ,\ \therefore\ \theta=60^\circ.$$

Here $\cos\theta$ being positive, θ may be in the 1st or 4th quadrant. In the fourth quadrant $\cos\theta=\cos(360^\circ-\theta)=\cos(360^\circ-60^\circ)=\cos 300^\circ$, so $\theta=300^\circ$. But this value of θ does not satisfy the given equation. Hence, the only value of θ is 60. Here 300° is an **extraneous root** of the equation.

Ex. 13. Find the values of θ, lying between 0° and 360° satisfying the equation $3(\sec^2\theta+\tan^2\theta)=5$.

$$3(\sec^2\theta+\tan^2\theta)=5,$$

$$\text{or,}\quad 3(1+\tan^2\theta+\tan^2\theta)=5,\quad \text{or,}\quad 3+6\tan^2\theta=5,$$

$$\text{or,}\quad 6\tan^2\theta=2,\quad \text{or,}\quad \tan^2\theta=\tfrac{1}{3},\ \therefore\ \tan\theta=\pm\frac{1}{\sqrt{3}}.$$

Now, (i) if $\tan\theta=\frac{1}{\sqrt{3}}$, then $\tan\theta$ being positive, θ will be either in the 1st or in the 3rd quadrant.

$$\text{Hence, } \tan\theta=\frac{1}{\sqrt{3}}=\tan 30^\circ \text{ or } \tan(180^\circ+30^\circ),$$

$$\therefore\quad \theta=30^\circ \text{ or } 210^\circ.$$

Again (ii) if $\tan\theta=-\frac{1}{\sqrt{3}}$, then $\tan\theta$ being negative, θ will lie either in the 2nd or in the 4th quadrant.

$$\therefore\quad \tan\theta=-\frac{1}{\sqrt{3}}=-\tan 30^\circ=\tan(180^\circ-30^\circ)$$

$$\text{or,}=\tan(360^\circ-30^\circ)$$

$$\therefore\quad \theta=180^\circ-30^\circ,\quad \text{or,}\quad \theta=360^\circ-30^\circ$$

$$\therefore\quad \theta=150^\circ \text{ or } 330^\circ.$$

$\therefore$ the values of θ may be 30°, 150°, 210°, 330°.

Ex. 14. Evaluate $\sin\{n\pi+(-1)^n\frac{\pi}{6}\}$, where n is any integer.

Here n is an even or odd integer.

(i) If n be an even integer, let $n=2p$ (p being any integer),

$$\therefore\ \sin\left\{n\pi+(-1)^n\frac{\pi}{6}\right\}=\sin\left\{2p\pi+(-1)^{2p}\frac{\pi}{6}\right\}$$

$$=\sin\left(2p\pi+\frac{\pi}{6}\right) \qquad [\because 2p \text{ is even, } \therefore (-1)^{2p}=1]$$

$$=\sin\frac{\pi}{6}=\frac{1}{2}.$$

(ii) If n be an odd integer, let $n=2p+1$ (where p is any integer).

$$\therefore\ \sin\left\{n\pi+(-1)^n\frac{\pi}{6}\right\}=\sin\left\{(2p+1)\pi+(-1)^{2p+1}\frac{\pi}{6}\right\}$$

$$=\sin\left\{(2p+1)\pi-\frac{\pi}{6}\right\}=\sin\left\{2p\pi+\left(\pi-\frac{\pi}{6}\right)\right\}$$

$$=\sin\left(\pi-\frac{\pi}{6}\right)=\sin\frac{\pi}{6}=\frac{1}{2}.$$

[**N. B.** Since $2p+1$ is odd, $(-1)^{2p+1}=-1$]

✓**Ex. 15.** If ABCD is a cyclic quadrilateral, show that $\tan A+\tan B+\tan C+\tan D=0$.

$\because$ the sum of a pair of opposite angles of a cyclic quadrilateral = 2 right angles = 180°,

$\therefore$ $A+C=180°$, or $A=180°-C$,

and $B+D=180°$, or $B=180°-D$.

Now, $\tan A+\tan B+\tan C+\tan D$

$$=\tan(180°-C)+\tan(180°-D)+\tan C+\tan D$$

$$=-\tan C-\tan D+\tan C+\tan D=0.$$

Exercise 1

Find the smallest positive co-terminal angle and the value of the expression :—

1. $\cos 420^\circ$ **2.** $\tan(-315^\circ)$ **3.** $\sec \frac{25\pi}{6}$.

Find the value of :—

4. $\cot 585^\circ$ **5.** $\sin(-1215^\circ)$ **6.** $\cot\left(-\frac{5\pi}{6}\right)$

7. $\sin 960^\circ$ **8.** $\tan 675^\circ$ **9.** $\sec(-1575^\circ)$

10. $\operatorname{cosec}(-1470^\circ)$ **11.** $\tan 1200^\circ$

12. $\sec\left(\frac{3\pi}{2}+\frac{\pi}{3}\right)$ **13.** $\sin(-1125^\circ)+\cos(-1125^\circ)$

14. $\dfrac{\cot 315^\circ-\cos(-240^\circ)}{\cot 990^\circ+\cos\left(\frac{16\pi}{3}\right)}$

Prove that :—

15. $\cos(A-270^\circ)=-\sin A$.

16. $\sin(780^\circ)\cos(390^\circ)-\sin(330^\circ)\cos(-300^\circ)=1$.

17. $\cos(n\pi+\theta)=(-1)^n\cos\theta$.

18. $\dfrac{\sin\left(\frac{\pi}{2}-\theta\right)\sin\left(\frac{3\pi}{2}-\theta\right)\cot\left(\frac{\pi}{2}+\theta\right)}{\sin\theta\cos\theta}=1$.

19. If $\sec\frac{2\pi}{3}=-2$, find $\sin\frac{2\pi}{3}$.

20. If $\cot\theta=-\frac{8}{15}$, find $\sin\theta$.

21. If n be an even integer, evaluate $\sin\theta+\sin(\pi+\theta)+\sin(2\pi+\theta)+\cdots$ to n terms.

Express in terms of ratios of positive angles less than 45° :—

22. $\tan 142^\circ$ **23.** $\cos(-930^\circ)$ **24.** $\sin\frac{17\pi}{9}$.

25. Find all the angles numerically less than 360° which satisfy the equation $\tan\theta=-\sqrt{3}$.

26. What values between 0° and 360° may θ have if $\sin\theta=\dfrac{\sqrt{3}}{2}$?

Solve for θ, giving all possible values, when $0^\circ<\theta<360^\circ$:—

27. $\cot\theta+\tan\theta=2\sec\theta$. **28.** $\sec\theta=\dfrac{2}{\sqrt{3}}$.

29. $2\sin^2\theta+\cos\theta=2$. **30.** $\sin\theta+\sqrt{3}\cos\theta=2$.

31. $2(\sec^2\theta+\sin^2\theta)=5$.

32. If n is any integer, find the values of :—

(i) $\cos\left\{n\pi+(-1)^n\dfrac{\pi}{3}\right\}$ (ii) $\sin\left\{\dfrac{n\pi}{2}+(-1)^n\dfrac{\pi}{4}\right\}$

(iii) $\tan\left\{n\pi+(-1)^n\dfrac{\pi}{4}\right\}$.

33. Find the values of A when,

(i) $\sin A=-\dfrac{\sqrt{3}}{2}$ and A lies between 180° and 270°.

(ii) $\tan A=-\dfrac{1}{\sqrt{3}}$ and A lies between 270° and 360°.

(iii) $\sec A=\sqrt{2}$ and $360^\circ<A<450^\circ$.

34. An angle θ lies between 270° and 360°, and $\sin\theta=-\frac{1}{2}$, find $\sec\theta$.

35. If $\cot\theta=\frac{3}{4}$ and $\sin\theta$ is negative, find the value of

$$\frac{\cot(-\theta)+\operatorname{cosec}\theta}{\cos\theta+\sin(-\theta)}.$$

36. Simplify and evaluate, when $\theta=240^\circ$

(i) $$\frac{\sin(\pi-\theta)}{\tan(\pi+\theta)}\cdot\frac{\cot\left(\dfrac{\pi}{2}-\theta\right)}{\tan\left(\dfrac{\pi}{2}+\theta\right)}\cdot\frac{\cos(2\pi-\theta)}{\sin(-\theta)}$$

(ii) $$\frac{\sin(-\theta)}{\sin(\pi+\theta)}-\frac{\tan\left(\dfrac{\pi}{2}+\theta\right)}{\cot\theta}+\frac{\cos\theta}{\sin\left(\dfrac{\pi}{2}+\theta\right)}.$$

37. ABCD is a quadrilateral ; prove that $\tan \frac{1}{2}(A+B)+\tan \frac{1}{2}(C+D)=0$.

38. If ABC be a triangle, find the value of

$$\frac{\sin(A+B)+\sin(B+C)+\sin(C+A)}{\cos\left(\frac{3\pi}{2}-C\right)+\sin(2\pi-A)+\sin(\pi+B)}.$$

Compound Angles

(Addition and subtraction Formulas)

13. Compound angle : An angle formed by the algebraic sum of two or more angles is called a **compound angle.**

Thus, $A+B$, $A-B$, $A+B+C$, $A+B-C$, etc. are compound angles. We prove below some fundamental properties relating to the functions of compound angles.

14. Theorem 1. To prove geometrically that

(1) $\sin(A+B)=\sin A\cos B+\cos A\sin B$

and (2) $\cos(A+B)=\cos A\cos B-\sin A\sin B$

where A and B are positive acute angles and $A+B<90°$.

(1) Let the revolving line OT starting from its original position OT revolve in the positive direction and trace out $\angle TOT_1=A$ and then let it revolve further in the same direction and describe an angle $T_1OT_2=B$.

Then $\angle TOT_2=A+B$.

Cons. Take any pt. P in OT_2, the terminal position of the revolving line OT and from P draw PQ and PM perpendicular to OT and OT_1

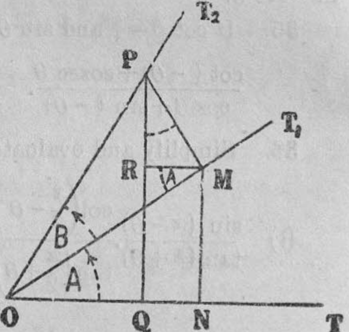

Fig. 9

respectively. From M draw MN and MR perpendicular to OT and PQ respectively.

Proof. $\because$ MR and TQ are both perpendicular to PQ,

$\therefore$ MR $\parallel$ TQ, $\therefore$ $\angle OMR =$ alt. $\angle MON = A$.

Again, $\angle OMR + \angle PMR = 1$ rt. angle $= \angle PMR + \angle RPM$

$\therefore$ $\angle MPR = \angle OMR = A$.

Now from the right-angled $\triangle POQ$, we have

$$\sin (A+B) = \sin \angle POQ = \frac{PQ}{OP} = \frac{RQ+PR}{OP}$$

$$= \frac{MN+PR}{OP} \quad [\because \text{ in rectangle } MNQR,\ RQ = MN]$$

$$= \frac{MN}{OP} + \frac{PR}{OP} = \frac{MN}{OM}.\frac{OM}{OP} + \frac{PR}{PM}.\frac{PM}{OP}$$

$$= \sin A \cos B + \cos A \sin B.$$

(2) [Same as (1) up to the construction]

Proof. RM $\parallel$ QN or OT both being perpendicular to PQ.

$\therefore$ $\angle OMR =$ alt. $\angle MON = A$.

Again, $\angle OMR + \angle PMR = 1$ rt. angle $= \angle PMR + \angle RPM$.

$\therefore$ $\angle RPM = \angle OMR = A$.

RQ $\parallel$ MN, both being perpendicular to OT.

$$\text{Now, } \cos (A+B) = \cos \angle POQ = \frac{OQ}{OP} = \frac{ON-QN}{OP}$$

$$= \frac{ON-MR}{OP} \quad [\because \text{ in rectangle } MNQR,\ MR = QN.]$$

$$= \frac{ON}{OP} - \frac{MR}{OP} = \frac{ON}{OM}.\frac{OM}{OP} - \frac{MR}{PM}.\frac{PM}{OP}$$

$$= \cos A \cos B - \sin A \sin B.$$

[**N. B.** If in the above theorem A and B are positive acute angles but their sum is not less than 90°, then the figure will be as given here.

In both the above cases the angles A and B are acute angles. The same proof will hold good, when A and B are angles of any size, if proper signs (positive or negative) of the quantities involved be assigned.

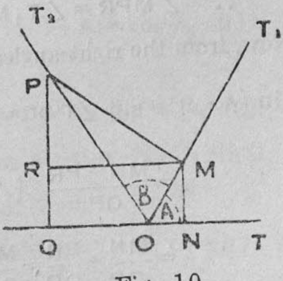

Fig. 10

15. Theorem 2. To prove that

(1) **sin** $(A-B)=$ **sin** A **cos** $B-$ **cos** A **sin** B,

and (2) **cos** $(A-B)=$ **cos** A **cos** $B+$ **sin** A **sin** B,

where A and B are positive acute angles and $A>B$.

(1) Let the revolving line OT starting from its original position OT revolve in the positive direction and describe an angle $TOT_1=A$, and let it then revolve back (*i. e.*, in the negative direction) and describe an angle $T_1OT_2=B$.

Then $\angle TOT_2=A-B$.

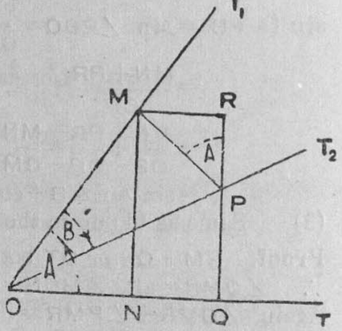

Fig. 11

Cons. On OT_2, the final position of the revolving line, take a pt. P and from P, draw PQ and PM perpendicular to OT and OT_1. Again, from M draw MN and MR perpendicular to OT and QP produced.

Proof. $\because$ OQ and MR are both perpendicular to RQ,

$\therefore$ $MR \parallel OQ$,

$\therefore$ $\angle T_1MR=$ corresponding $\angle MON=A$.

Again, $\angle MPR+\angle PMR=1$ rt. angle $=\angle PMR+\angle T_1MR$

$\therefore$ $\angle MPR=\angle T_1MR=A$.

Now, from the right-angled $\triangle POQ$ we have

$$\textbf{sin }(A-B)=\sin\angle POQ=\frac{PQ}{OP}=\frac{RQ-PR}{OP}$$

$$=\frac{MN-PR}{OP}\quad [\because \text{ in rectangle } MNQR\ RQ=MN]$$

$$=\frac{MN}{OP}-\frac{PR}{OP}=\frac{MN}{OM}\cdot\frac{OM}{OP}-\frac{PR}{PM}\cdot\frac{PM}{OP}$$

$$=\sin A\cos B-\cos A\sin B.$$

(2) [Same as in (1) up to the construction]

Proof. Both MR, OQ are perpendicular to PQ, ∴ MR ∥ OQ.

∴ $\angle T_1MR =$ alt. $\angle MON = A$.

Again, $\angle PMR + \angle MPR = 1$ rt. angle $= \angle PMR + \angle T_1MR$,

∴ $\angle MPR = \angle T_1MR = A$.

MN ∥ RQ, both being perpendicular to OT.

Now, $\cos(A-B) = \cos\angle POQ = \dfrac{OQ}{OP} = \dfrac{ON+NQ}{OP}$

$= \dfrac{ON+MR}{OP}$ [∵ in rectangle MNQR, $NQ = MR$]

$= \dfrac{ON}{OP} + \dfrac{MR}{OP} = \dfrac{ON}{OM}\cdot\dfrac{OM}{OP} + \dfrac{MR}{PM}\cdot\dfrac{PM}{OP}$

$= \cos A \cos B + \sin A \sin B$.

[**N. B.** In the above theorem A and B are assumed to be positive acute angles, $A+B$ less than a right angle and $A-B$ as positive. If A and B be angles of any magnitude, the theorem can be proved as before by drawing a figure and assigning proper signs to the quantities involved.]

The theorem can also be proved, as shown below, without drawing any figure.

Let $X = 90^\circ + A$, then $\sin X = \sin(90^\circ + A) = \cos A$, and

$\cos X = \cos(90^\circ + A) = -\sin A$.

∴ $\sin(X+B) = \sin\{(90^\circ + A) + B\} = \sin\{90^\circ + (A+B)\}$

$= \cos(A+B) = \cos A \cos B - \sin A \sin B$

$= \sin X \cos B + \cos X \sin B$.

Again, $\cos(X+B) = \cos\{90^\circ + (A+B)\} = -\sin(A+B)$

$= -\sin A \cos B - \cos A \sin B$

$= \cos X \cos B - \sin X \sin B$.

Similarly the theorem can be proved by putting $X = 90^\circ + B$ and also by putting $X_1 = 90^\circ + X$.

Again, let $X = -A$

$$\begin{aligned} \therefore \sin(X+B) &= \sin(-A+B) = \sin\{-(A-B)\} = -\sin(A-B) \\ &= -(\sin A \cos B - \cos A \sin B) \\ &= -\sin A \cos B + \cos A \sin B \\ &= \sin(-A) \cos B + \cos(-A) \sin B \\ &= \sin X \cos B + \cos X \sin B. \end{aligned}$$

It can similarly be proved that the theorem is true for all values of A and B.

Hence, the above two theorems are always true. They are called Addition Theorems or Subtraction Theorems.

16. Proof of some corollaries

Cor. 1. Prove that $\sin(A+B)\sin(A-B)$

$$= \sin^2 A - \sin^2 B \ldots\ldots (i)$$
$$= \cos^2 B - \cos^2 A \ldots\ldots (ii)$$

Proof. $\sin(A+B)\sin(A-B)$

$$\begin{aligned} &= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B) \\ &= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B \\ &= \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B \\ &= \sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B \\ &= \sin^2 A - \sin^2 B \ldots\ldots (i) \text{ [proved]} \\ &= (1 - \cos^2 A) - (1 - \cos^2 B) \\ &= \cos^2 B - \cos^2 A \ldots\ldots (ii). \text{ [proved].} \end{aligned}$$

Cor. 2. Prove that $\cos(A+B)\cos(A-B)$

$$= \cos^2 A - \sin^2 B \ldots (i)$$
$$= \cos^2 B - \sin^2 A \ldots (ii)$$

Proof. $\cos(A+B)\cos(A-B)$

$$\begin{aligned} &= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B) \\ &= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B \\ &= \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A)\sin^2 B \\ &= \cos^2 A - \sin^2 B \ldots\ldots (i) \\ &= (1 - \sin^2 A) - (1 - \cos^2 B) \\ &= 1 - \sin^2 A - 1 + \cos^2 B = \cos^2 B - \sin^2 A \ldots\ldots (ii). \end{aligned}$$

Cor. 3. Show that (i) $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

(ii) $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$.

Proof : (i) $\tan(A+B)=\dfrac{\sin(A+B)}{\cos(A+B)}$

$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

Dividing the numerator and the denominator on the R. H. S. by $\cos A\cos B$, we have

$$\tan(A+B)=\frac{\dfrac{\sin A\cos B}{\cos A\cos B}+\dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}-\dfrac{\sin A\sin B}{\cos A\cos B}}=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

(ii) $\tan(A-B)=\dfrac{\sin(A-B)}{\cos(A-B)}=\dfrac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B}$

Dividing the numerator and the denominator on the R. H. S. by $\cos A\cos B$, we have

$$\tan(A-B)=\frac{\dfrac{\sin A\cos B}{\cos A\cos B}-\dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}+\dfrac{\sin A\sin B}{\cos A\cos B}}=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

Cor. 4. Prove that (i) $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot B+\cot A}$,

(ii) $\cot(A-B)=\dfrac{\cot A\cot B+1}{\cot B-\cot A}$.

Proof : (i) $\cot(A+B)=\dfrac{\cos(A+B)}{\sin(A+B)}$

$$=\frac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}$$

Dividing the numerator and the denominator on the R. H. S. by sin A sin B, we have

$$\cot(A+B)=\frac{\dfrac{\cos A\cos B}{\sin A\sin B}-\dfrac{\sin A\sin B}{\sin A\sin B}}{\dfrac{\sin A\cos B}{\sin A\sin B}+\dfrac{\cos A\sin B}{\sin A\sin B}}=\frac{\cot A\cot B-1}{\cot B+\cot A}.$$

(ii) $\cot(A-B)=\dfrac{\cos(A-B)}{\sin(A-B)}=\dfrac{\cos A\cos B+\sin A\sin B}{\sin A\cos B-\cos A\sin B}$

Dividing the numerator and the denominator on the R. H. S. by sin A sin B, we have

$$\cot(A-B)=\frac{\dfrac{\cos A\cos B}{\sin A\sin B}+\dfrac{\sin A\sin B}{\sin A\sin B}}{\dfrac{\sin A\cos B}{\sin A\sin B}-\dfrac{\cos A\sin B}{\sin A\sin B}}=\frac{\cot A\cot B+1}{\cot B-\cot A}.$$

Cor. 5. Find the expansion of

(i) **sin** $(A+B+C)$

(ii) **cos** $(A+B+C)$

(iii) **tan** $(A+B+C)$.

Proof : (i) $\sin(A+B+C)=\sin\{(A+B)+C\}$

$=\sin(A+B)\cos C+\cos(A+B)\sin C$

$=(\sin A\cos B+\cos A\sin B)\cos C$

$+(\cos A\cos B-\sin A\sin B)\sin C$

$=\sin A\cos B\cos C+\sin B\cos C\cos A$

$+\sin C\cos A\cos B-\sin A\sin B\sin C.$

[N. B. Sin $(A+B+C)$ can be written in the form $\cos A\cos B\cos C(\tan A+\tan B+\tan C-\tan A\tan B\tan C)$].

(ii) $\cos(A+B+C)=\cos\{(A+B)+C\}$

$=\cos(A+B)\cos C-\sin(A+B)\sin C$

$=(\cos A\cos B-\sin A\sin B)\cos C$

$-(\sin A\cos B+\cos A\sin B)\sin C$

$$= \cos A \cos B \cos C - \sin A \sin B \cos C$$
$$- \sin A \sin C \cos B - \sin B \sin C \cos A.$$

[**N. B.** The expansion of $\cos(A+B+C)$ can be written as $\cos A \cos B \cos C(1 - \tan A \tan B - \tan A \tan C - \tan B \tan C)$.]

(iii) $\tan(A+B+C) = \tan\{(A+B)+C\}$

$$= \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C}$$

$$= \frac{\dfrac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \dfrac{\tan A + \tan B}{1 - \tan A \tan B}\tan C}$$

$$= \frac{\dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B}}{\dfrac{1 - \tan A \tan B - (\tan A + \tan B)\tan C}{1 - \tan A \tan B}}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan A \tan C - \tan B \tan C}.$$

[**N. B.** (1) This should be remembered. The above expansion of $\tan(A+B+C)$ can be had by writing $\tan(A+B+C) = \dfrac{\sin(A+B+C)}{\cos(A+B+C)}$ and then expanding the numerator and the denominator and dividing each by $\cos A \cos B \cos C$.

(2) Similarly the trigonometrical functions or expansions of the sum of 4, 5 or more angles can be found.]

Examples (2)

Ex. 1. Find the values of $\sin 75^\circ$, $\cos 75^\circ$ and $\tan 75^\circ$.

sin 75 $= \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

$$\cos 75^\circ = \cos(45^\circ + 30^\circ) = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

$$\tan 75^\circ = \tan(45^\circ + 30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \times \frac{1}{\sqrt{3}}} = \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$= \frac{(\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{4+2\sqrt{3}}{2} = 2 + \sqrt{3}.$$

Ex. 2. Find the value of tan 15°.

$$\tan 15^\circ = \tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \times \frac{1}{\sqrt{3}}} = \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$= \frac{4-2\sqrt{3}}{2} = 2 - \sqrt{3}.$$

Ex. 3. Find the value of sin (A − B), when $\sin A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$.

$$\because \sin A = \tfrac{3}{5}, \quad \therefore \cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \tfrac{9}{25}}$$

$$= \sqrt{\tfrac{16}{25}} = \tfrac{4}{5}.$$

$$\because \sin B = \tfrac{5}{13}, \quad \therefore \cos B = \sqrt{1 - \sin^2 B}$$

$$= \sqrt{1 - \tfrac{25}{169}} = \sqrt{\tfrac{144}{169}} = \tfrac{12}{13}.$$

Now, $\sin(A - B) = \sin A \cos B - \cos A \sin B = \frac{3}{5} \times \frac{12}{13} - \frac{4}{5} \times \frac{5}{13}$

$$= \tfrac{36}{65} - \tfrac{20}{65} = \tfrac{16}{65}.$$

[N. B. Actually $\cos A = \pm\frac{4}{5}$ and $\cos B = \pm\frac{12}{13}$; but in such cases we generally take the positive square root.]

Ex. 4. Prove that $\tan(45^\circ+A)=\dfrac{1+\tan A}{1-\tan A}$.

$$\tan(45^\circ+A)=\frac{\tan 45^\circ+\tan A}{1-\tan 45^\circ \tan A}$$

$$=\frac{1+\tan A}{1-\tan A}\quad [\because \tan 45^\circ=1].$$

Ex. 5. Prove that $\tan(A+B)\tan(A-B)=\dfrac{\sin^2 A-\sin^2 B}{\cos^2 A-\sin^2 B}$.

[C. U. '44]

$$\text{The L. H. S.}=\frac{\sin(A+B).\sin(A-B)}{\cos(A+B).\cos(A-B)}$$

$$=\frac{\sin^2 A-\sin^2 B}{\cos^2 A-\sin^2 B}\quad [\text{see cor. } 1,\ 2.]$$

Ex. 6. Show that $\dfrac{\cos 6^\circ+\sin 6^\circ}{\cos 6^\circ-\sin 6^\circ}=\tan 51^\circ$.

$$\cos 6^\circ+\sin 6^\circ=\cos(51^\circ-45^\circ)+\sin(51^\circ-45^\circ)$$

$$=\cos 51^\circ.\frac{1}{\sqrt{2}}+\sin 51^\circ.\frac{1}{\sqrt{2}}+\sin 51^\circ.\frac{1}{\sqrt{2}}-\cos 51^\circ.\frac{1}{\sqrt{2}}$$

$$=2\sin 51^\circ.\frac{1}{\sqrt{2}}=\sqrt{2}\sin 51^\circ.$$

$$\cos 6^\circ-\sin 6^\circ=\cos(51^\circ-45^\circ)-\sin(51^\circ-45^\circ)$$

$$=2\cos 51^\circ.\frac{1}{\sqrt{2}}=\sqrt{2}\cos 51^\circ.$$

$$\therefore\quad \frac{\cos 6^\circ+\sin 6^\circ}{\cos 6^\circ-\sin 6^\circ}=\frac{\sqrt{2}\sin 51^\circ}{\sqrt{2}\cos 51^\circ}=\tan 51^\circ.$$

Ex. 7. Prove that $\cot A-\cot 2A=\operatorname{cosec} 2A$.

$$\cot A-\cot 2A=\frac{\cos A}{\sin A}-\frac{\cos 2A}{\sin 2A}$$

$$=\frac{\sin 2A\cos A-\sin A\cos 2A}{\sin A\sin 2A}=\frac{\sin(2A-A)}{\sin A\sin 2A}$$

$$=\frac{\sin A}{\sin A\sin 2A}=\frac{1}{\sin 2A}=\operatorname{cosec} 2A.$$

Ex. 8. Prove the identity

$$\cos^2 A+\cos^2\left(A+\frac{\pi}{3}\right)+\cos^2\left(A-\frac{\pi}{3}\right)=\frac{3}{2}.$$ [C. U. '33]

$$\text{L. H. S.}=\cos^2 A+\{\cos(A+60^\circ)\}^2+\{\cos(A-60^\circ)\}^2$$
$$=\cos^2 A+(\cos A\cos 60^\circ-\sin A\sin 60^\circ)^2+(\cos A\cos 60^\circ+\sin A\sin 60^\circ)^2$$
$$=\cos^2 A+\left(\tfrac{1}{2}\cos A-\frac{\sqrt{3}}{2}\sin A\right)^2+\left(\tfrac{1}{2}\cos A+\frac{\sqrt{3}}{2}\sin A\right)^2$$
$$=\cos^2 A+2\left(\tfrac{1}{4}\cos^2 A+\tfrac{3}{4}\sin^2 A\right)$$
$$=\cos^2 A+\tfrac{1}{2}\cos^2 A+\tfrac{3}{2}\sin^2 A=\tfrac{3}{2}\cos^2 A+\tfrac{3}{2}\sin^2 A$$
$$=\tfrac{3}{2}(\cos^2 A+\sin^2 A)=\tfrac{3}{2}\times 1=\tfrac{3}{2}.$$

Ex. 9. Prove that $\tan 10^\circ+\tan 35^\circ+\tan 10^\circ\tan 35^\circ=1.$

$$\because\ \frac{\tan 10^\circ+\tan 35^\circ}{1-\tan 10^\circ\tan 35^\circ}=\tan(10^\circ+35^\circ)=\tan 45^\circ=1,$$
$$\therefore\ \tan 10^\circ+\tan 35^\circ=1-\tan 10^\circ\tan 35^\circ,$$
$$\therefore\ \tan 10^\circ+\tan 35^\circ+\tan 10^\circ\tan 35^\circ=1.$$

✓**Ex. 10.** If $\cos(A+B)\sin(C+D)=\cos(A-B)\sin(C-D)$, show that $\cot A.\cot B.\cot C=\cot D.$ [C. U. '30]

$$\because\ \cos(A+B)\sin(C+D)=\cos(A-B)\sin(C-D),$$
$$\therefore\ \frac{\cos(A+B)}{\cos(A-B)}=\frac{\sin(C-D)}{\sin(C+D)},$$
Or, $$\frac{\cos(A+B)+\cos(A-B)}{\cos(A+B)-\cos(A-B)}=\frac{\sin(C-D)+\sin(C+D)}{\sin(C-D)-\sin(C+D)}$$ [By Comp. & Div.]

Or, $$\frac{2\cos A\cos B}{-2\sin A\sin B}=\frac{2\sin C\cos D}{-2\cos C\sin D},$$

Or, $$\frac{\cos A\cos B}{\sin A\sin B}=\frac{\sin C\cos D}{\cos C\sin D},$$

Or, $$\cot A\cot B=\tan C\cot D=\frac{\cot D}{\cot C}\left[\because\ \tan C=\frac{1}{\cot C}\right]$$

$$\therefore\ \cot A\cot B\cot C=\cot D.$$

Ex. 11. If $\sin(A+B) = n\sin(A-B)$ and if $n \neq -1$, shew that $\cot A = \frac{n-1}{n+1}\cot B$.

$\because\ \sin(A+B) = n\sin(A-B)$,

$\therefore\ \sin A\cos B + \cos A\sin B = n(\sin A\cos B - \cos A\sin B)$,

Or, $\cos A\sin B + n\cos A\sin B$

$= n\sin A\cos B - \sin A\cos B$ [by transposition]

Or, $\cos A\ \sin B(1+n) = \sin A\cos B(n-1)$,

Or, $\frac{\cos A\sin B}{\sin A\cos B} = \frac{n-1}{n+1}$, *or,* $\cot A\tan B = \frac{n-1}{n+1}$,

$\therefore$ *Or,* $\cot A \times \frac{1}{\cot B} = \frac{n-1}{n+1}$, $\therefore\ \cot A = \frac{n-1}{n+1}\cot B$.

Ex. 12. Show that if an angle α be divided into two parts so that the ratio of the tangents of the parts is λ, the difference x between the parts is given by the equation $\sin x = \frac{\lambda-1}{\lambda+1}\sin\alpha$. [A. U. '45]

Let m and n be the two parts of the angle α.

$\therefore$ from the given condition we have $\frac{\tan m}{\tan n} = \lambda \cdots\cdots (1)$

$m - n = x \cdots\cdots (2)$, and $m + n = \alpha \cdots\cdots (3)$

$$\text{Now, } \frac{\lambda-1}{\lambda+1}\sin\alpha = \frac{\frac{\tan m}{\tan n} - 1}{\frac{\tan m}{\tan n} + 1}\cdot\sin\alpha = \frac{\tan m - \tan n}{\tan m + \tan n}\sin$$

$$= \frac{\frac{\sin m}{\cos m} - \frac{\sin n}{\cos n}}{\frac{\sin m}{\cos m} + \frac{\sin n}{\cos n}}\cdot\sin(m+n)\ [\because \alpha = m+n]$$

$$= \frac{\sin m\cos n - \cos m\sin n}{\sin m\cos n + \cos m\sin n}\cdot\sin(m+n)$$

$$= \frac{\sin(m-n)}{\sin(m+n)}\cdot\sin(m+n) = \sin(m-n)$$

$$= \sin x.$$

Ex. 13. If $A+B=C$, prove that

$$\cos^2 A+\cos^2 B-2\cos A\cos B\cos C=\sin^2 C.$$

L.H.S. $=\cos^2 A+\cos^2 B-2\cos A\cos B\cos(A+B)$ $[\because C=A+B]$

$$=\cos^2 A+\cos^2 B-2\cos A\cos B\times(\cos A\cos B-\sin A\sin B)$$

$$=\cos^2 A+\cos^2 B-2\cos^2 A\cos^2 B+2\cos A\cos B\sin A\sin B$$

$$=\cos^2 A-\cos^2 A\cos^2 B+\cos^2 B-\cos^2 A\cos^2 B+2\cos A\cos B\sin A\sin B$$

$$=\cos^2 A(1-\cos^2 B)+\cos^2 B(1-\cos^2 A)+2\cos A\cos B\sin A\sin B$$

$$=\cos^2 A\sin^2 B+\cos^2 B\sin^2 A+2\cos A\cos B\sin A\sin B$$

$$=(\cos A\sin B+\cos B\sin A)^2=\{\sin(A+B)\}^2=\sin^2 C.$$

Exercise 2

1. Find the value of (i) $\sin(-15^\circ)$, (ii) $\cot 15^\circ$ (iii) $\tan(-75^\circ)$.
2. Find the expansion of $\sin(A-B+C)$.
3. Find the value of $\sin 105^\circ+\cos 105^\circ+\cos\frac{\pi}{4}$.
4. If $\sin A=\frac{3}{5}$, $\cos B=\frac{12}{13}$, find the value of $\cos(A+B)$.
5. If $\sin B=\frac{4}{5}$ and $\cos A=\frac{8}{17}$, find the value of $\sec(A+B)$.
6. If $\tan A=\frac{2}{11}$ and $\cot B=\frac{24}{7}$, find $\cot(A-B)$.

Prove the following identities :—

7. $2\sin\left(\frac{\pi}{6}-\theta\right)=\cos\theta-\sqrt{3}\sin\theta$
8. $\dfrac{\sin(A-B)}{\sin A\sin B}=\cot B-\cot A$.

9. $\cos A+\cos(120^\circ+A)+\cos(120^\circ-A)=0$ [C. U. '53]

10. $\cos\left(\frac{\pi}{3}-\theta\right)\cos\left(\frac{\pi}{6}-\alpha\right)-\sin\left(\frac{\pi}{3}-\theta\right)\sin\left(\frac{\pi}{6}-\alpha\right)$
$=\sin(\theta+\alpha)$.

11. $2\cos(45^\circ+A)\cos(45^\circ-A)=\cos^2 A-\sin^2 A$.

12. $\tan\left(\frac{\pi}{4}+\theta\right)=\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$.

13. $\frac{\tan(2\alpha-\beta)+\tan\beta}{1-\tan(2\alpha-\beta)\tan\beta}=\tan 2\alpha$.

14. $\cos^2 A+\cos^2(120^\circ-A)+\cos^2(120^\circ+A)=\frac{3}{2}$.

15. $\cos 69^\circ 22'\cos 9^\circ 22'+\cos 80^\circ 38'\cos 20^\circ 38'=\frac{1}{2}$.

16. $\cos 38^\circ 15'\sin 68^\circ 15'-\cos 51^\circ 45'\sin 21^\circ 45'=\frac{1}{2}$.

17. $\sin 3\theta\cos\theta-\cos 3\theta\sin\theta=\sin 2\theta$.

18. $\cot 2A+\tan A=\operatorname{cosec} 2A$. [C. U. '47]

19. $1+\frac{\tan 2A}{\cot A}=\sec 2A$. 20. $\frac{\cos 9^\circ+\sin 9^\circ}{\cos 9^\circ-\sin 9^\circ}=\tan 54^\circ$.

21. $\tan(A+B)+\tan(A-B)=\frac{\sin 2A}{1-\sin^2 A-\sin^2 B}$.

✓22. $\sin(m+1)\theta.\sin(m-1)\theta+\cos(m+1)\theta\cos(m-1)\theta$
$=\cos 2\theta$.

✓23. $\cos\frac{1}{2}(\phi-\theta)-\sin\theta.\sin\frac{1}{2}(\phi+\theta)=\cos\theta.\cos\frac{1}{2}(\phi+\theta)$.

24. $\tan^2 A-\tan^2 B=\frac{\sin^2 A-\sin^2 B}{\cos^2 A\cos^2 B}$. [C. U. '36]

25. $\tan(\alpha+\beta)\tan(\alpha-\beta)=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\alpha-\sin^2\beta}$.

26. $\tan 27^\circ+\tan 18^\circ+\tan 18^\circ\tan 27^\circ=1$.

Simplify :—

27. $\frac{\sin(A-B)}{\sin A\sin B}+\frac{\sin(B-C)}{\sin B\sin C}+\frac{\sin(C-A)}{\sin C\sin A}$.

28. $1+\dfrac{\sin(A-B)}{\cos A\cos B}+\dfrac{\sin(B-C)}{\cos B\cos C}+\dfrac{\sin(C-A)}{\cos C\cos A}.$

✓29. If $\sin\alpha\sin\beta-\cos\alpha\cos\beta+1=0$,

show that $1+\cot\alpha\tan\beta=0$. [C. U. '39]

✓30. If $A+B+C=\pi$ and $\cos A=\cos B\cos C$, show that $\tan A=\tan B+\tan C$. [C. U. '42]

✓31. If $\tan\beta=\dfrac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$, show that

$\tan(\alpha-\beta)=(1-n)\tan\alpha$. [P. U. '50]

✓32. If A, B, C be the angles of a triangle, show that $\sin^2 C=\cos^2 A+\cos^2 B+2\cos A\cos B\cos C$.

[C. U. '30]

33. Prove geometrically the formula

$\cos(A+B)=\cos A\cos B-\sin A\sin B$ where A is an obtuse angle and B is acute. [C. U. '41]

Transformation of Products and Sums

17. Transformation of products into sums or differences

(1) We have already proved in Art. 14 and 15 that

$\sin A\cos B+\cos A\sin B=\sin(A+B)\cdots$(i)

$\sin A\cos B-\cos A\sin B=\sin(A-B)\cdots$(ii)

$\therefore$ (adding) $2\sin A\cos B=\sin(A+B)+\sin(A-B)$.

Again, subtracting (ii) from (i) we have

$2\cos A\sin B=\sin(A+B)-\sin(A-B)$.

Here we have the above two formulas for transforming the product of a **sine** and a **cosine** into the **sum** or **difference** of two sines.

(2) From art. 14 and 15 we have

$$\cos A \cos B - \sin A \sin B = \cos (A+B) \ldots \text{(iii)}$$

$$\cos A \cos B + \sin A \sin B = \cos (A-B) \ldots \text{(iv)}$$

$\therefore$ (adding) $2 \cos A \cos B = \cos (A+B) + \cos (A-B)$.

Again, subtracting (iii) from (iv) we have

$$2 \sin A \sin B = \cos (A-B) - \cos (A+B).$$

Here we have the above two formulas for transforming (i) the product of two cosines into the sum of two cosines and (ii) the product of two sines into the difference of two cosines.

These four formulas are very important. They are given below :

$$2 \sin A \cos B = \sin (A+B) + \sin (A-B) \cdots (1)$$

$$2 \cos A \sin B = \sin (A+B) - \sin (A-B) \cdots (2)$$

$$2 \cos A \cos B = \cos (A+B) + \cos (A-B) \ldots (3)$$

$$2 \sin A \sin B = \cos (A-B) - \cos (A+B) \ldots (4)$$

[**N. B.** The following will be helpful for remembering the formulas :

$2 \sin A \cos B = \sin (\text{sum}) + \sin (\text{difference})$

$2 \cos A \sin B = \sin (\text{sum}) - \sin (\text{difference})$

$2 \cos A \cos B = \cos (\text{sum}) + \cos (\text{difference})$

$2 \sin A \sin B = \cos (\text{difference}) - \cos (\text{sum})$

In the last formula it is to be noted that the 1st term is difference and the 2nd one is sum.]

Example 1. $2 \sin 11A \cos 5A$

$$= \sin (11A + 5A) + \sin (11A - 5A)$$

$$= \sin 16A + \sin 6A.$$

Example 2. $2 \cos 5\theta \sin 8\theta = \sin (5\theta + 8\theta) - \sin (5\theta - 8\theta)$

$$= \sin 13\theta - \sin (-3\theta)$$

$$= \sin 13\theta + \sin 3\theta.$$

Example 3. $2 \cos 75^\circ \cos 15^\circ$

$$= \cos (75^\circ + 15^\circ) + \cos (75^\circ - 15^\circ)$$
$$= \cos 90^\circ + \cos 60^\circ = 0 + \tfrac{1}{2} = \tfrac{1}{2}.$$

Example 4. $2 \sin \frac{5\theta}{2} \sin \frac{7\theta}{2} = \cos \left(\frac{5\theta}{2} - \frac{7\theta}{2}\right) - \cos \left(\frac{5\theta}{2} + \frac{7\theta}{2}\right)$

$$= \{\cos(-\theta) - \cos 6\theta\} = \cos \theta - \cos 6\theta.$$

18. Transformation of sums or differences into products

$\because$ $\sin (A+B) = \sin A \cos B + \cos A \sin B$

and $\sin (A-B) = \sin A \cos B - \cos A \sin B$

$\therefore$ (adding) $\sin (A+B) + \sin (A-B) = 2 \sin A \cos B$......(1)

Now, let $A+B=C$, and $A-B=D$;

$$\text{Then } A = \frac{C+D}{2}, \text{ and } B = \frac{C-D}{2}.$$

$\therefore$ from (1) we have $\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$ ...(I)

Again, $\because$ $\sin (A+B) - \sin (A-B) = 2 \cos A \sin B$...(2),

$\cos (A+B) + \cos (A-B) = 2 \cos A \cos B$...(3),

and $\cos (A+B) - \cos (A-B)$

$$= -\{\cos (A-B) - \cos (A+B)\}$$
$$= -2 \sin A \sin B$$
$$= 2 \sin A \sin (-B) \text{...(4)},$$

$\therefore$ putting the values of A and B in terms of C and D in (2), (3) and (4) we have

$$\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2} \text{...(II)}$$

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2} \text{......(III)}$$

and $\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}$......(IV)

The above formulas also are very important. They are given below :

$$\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\ldots\ldots(\text{I})$$

$$\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\ldots\ldots(\text{II})$$

$$\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}\ldots\ldots(\text{III})$$

$$\cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}\ldots\ldots(\text{IV})$$

[It is to be noted that in (IV) it is $\frac{D-C}{2}$ and not $\frac{C-D}{2}$

as in other formulas. The following statement helps to remember the formulas :

(i) Sum of two sines = 2 sin ($\frac{1}{2}$ sum) cos ($\frac{1}{2}$ diff.)
(ii) difference of two sines = 2 cos ($\frac{1}{2}$ sum) sin ($\frac{1}{2}$ diff.)
(iii) sum of two cosines = 2 cos ($\frac{1}{2}$ sum) cos ($\frac{1}{2}$ diff.)
(iv) difference of two cosines
= 2 sin ($\frac{1}{2}$ sum) sin ($\frac{1}{2}$ diff. reversed)].

Example 1. $\sin 8\theta+\sin 6\theta=2\sin\frac{8\theta+6\theta}{2}\cos\frac{8\theta-6\theta}{2}$

$$=2\sin 7\theta\cos\theta.$$

Example 2. $\sin 14\theta-\sin 8\theta=2\cos\frac{14\theta+8\theta}{2}\sin\frac{14\theta-8\theta}{2}$

$$=2\cos 11\theta\sin 3\theta.$$

Example 3. $\cos 2\theta+\cos 5\theta=2\cos\frac{2\theta+5\theta}{2}\cos\frac{2\theta-5\theta}{2}$

$$=2\cos\frac{7\theta}{2}\cos\left(-\frac{3\theta}{2}\right)$$

$$=2\cos\frac{7\theta}{2}\cos\frac{3\theta}{2}.$$

Example 4. $\cos 75^\circ - \cos 15^\circ = 2\sin\dfrac{75^\circ+15^\circ}{2}\sin\dfrac{15^\circ-75^\circ}{2}$

$$= 2\sin 45^\circ \sin(-30^\circ)$$

$$= -2\sin 45^\circ \sin 30^\circ$$

$$= -2\times\frac{1}{\sqrt{2}}\times\frac{1}{2} = -\frac{1}{\sqrt{2}}.$$

Examples (3)

Ex. 1. Prove that $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$.

[P. U. '42]

$$\text{L. H. S.} = \sin 20^\circ \sin 40^\circ.\frac{\sqrt{3}}{2}.\sin 80^\circ \left[\because \sin 60^\circ = \frac{\sqrt{3}}{2}\right]$$

$$= \frac{\sqrt{3}}{4}\sin 20^\circ.(2\sin 80^\circ \sin 40^\circ)$$

$$= \frac{\sqrt{3}}{4}\sin 20^\circ\{\cos(80^\circ-40^\circ)-\cos(80^\circ+40^\circ)\}$$

$$= \frac{\sqrt{3}}{4}\sin 20^\circ\{\cos 40^\circ - \cos 120^\circ\}$$

$$= \frac{\sqrt{3}}{4}\sin 20^\circ\{\cos 40^\circ - (-\tfrac{1}{2})\}\ [\because \cos 120^\circ = -\tfrac{1}{2}]$$

$$= \frac{\sqrt{3}}{4}\sin 20^\circ(\cos 40^\circ + \tfrac{1}{2})$$

$$= \frac{\sqrt{3}}{4}\sin 20^\circ \cos 40^\circ + \frac{\sqrt{3}}{8}\sin 20^\circ$$

$$= \frac{\sqrt{3}}{8}\times 2\sin 20^\circ \cos 40^\circ + \frac{\sqrt{3}}{8}\sin 20^\circ$$

$$= \frac{\sqrt{3}}{8}\{\sin(20^\circ+40^\circ)+\sin(20^\circ-40^\circ)\} + \frac{\sqrt{3}}{8}\sin 20^\circ$$

$$= \frac{\sqrt{3}}{8}\{\sin 60^\circ + \sin(-20^\circ)\} + \frac{\sqrt{3}}{8}\sin 20^\circ$$

$$=\frac{\sqrt{3}}{8}\left(\frac{\sqrt{3}}{2}-\sin 20^\circ\right)+\frac{\sqrt{3}}{8}\sin 20^\circ$$

$$=\frac{3}{16}-\frac{\sqrt{3}}{8}\sin 20^\circ+\frac{\sqrt{3}}{8}\sin 20^\circ=\frac{3}{16}.$$

Ex. 2. Find the value of $\cos 80^\circ+\cos 40^\circ-\cos 20^\circ$.

The given exp. $=2\cos\dfrac{80^\circ+40^\circ}{2}\cos\dfrac{80^\circ-40^\circ}{2}-\cos 20^\circ$

$$=2\cos 60^\circ\cos 20^\circ-\cos 20^\circ=2\times\tfrac{1}{2}\times\cos 20^\circ-\cos 20^\circ$$

$$=\cos 20^\circ-\cos 20^\circ=0.$$

Ex. 3. Show that $\cos 5^\circ-\sin 25^\circ=\sin 35^\circ$.

$\because\ \sin 25^\circ=\cos(90^\circ-25^\circ)$,

$\therefore\ \cos 5^\circ-\sin 25^\circ=\cos 5^\circ-\cos(90^\circ-25^\circ)$

$$=\cos 5^\circ-\cos 65^\circ=2\sin\frac{5^\circ+65^\circ}{2}\sin\frac{65^\circ-5^\circ}{2}$$

$$=2\sin 35^\circ\sin 30^\circ=2\sin 35^\circ\times\tfrac{1}{2}=\sin 35^\circ.$$

Ex. 4. Prove that $\cos^2 A+\cos^2(60^\circ+A)+\cos^2(60^\circ-A)=\frac{3}{2}$.

$\because\ 2\cos^2 A=\cos^2 A+\cos^2 A=\cos^2 A+1-\sin^2 A$

$$=\cos A.\cos A-\sin A.\sin A+1$$

$$=\cos(A+A)+1=\cos 2A+1,$$

$\therefore$ L. H. S. $=\frac{1}{2}(1+\cos 2A)+\frac{1}{2}\{\cos(120^\circ+2A)+1\}$
$+\frac{1}{2}\{\cos(120^\circ-2A)+1\}$

$$=\tfrac{1}{2}\{3+\cos 2A+\cos(120^\circ+2A)+\cos(120^\circ-2A)\}$$

$$=\tfrac{1}{2}(3+\cos 2A+2\cos 120^\circ\cos 2A)$$

$$=\tfrac{1}{2}(3+\cos 2A+2\times-\tfrac{1}{2}\times\cos 2A)$$

$$=\tfrac{1}{2}(3+\cos 2A-\cos 2A)=\tfrac{3}{2}.$$

Ex. 5. Show that $\dfrac{\sin A-\sin B}{\cos B-\cos A}=\cot\dfrac{A+B}{2}$.

$$\text{L. H. S.}=\frac{2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}}{2\sin\dfrac{B+A}{2}\sin\dfrac{A-B}{2}}=\frac{\cos\dfrac{A+B}{2}}{\sin\dfrac{A+B}{2}}=\cot\frac{A+B}{2}.$$

✓**Ex. 6.** Prove that $\dfrac{\sin\theta+\sin 2\theta+\sin 4\theta+\sin 5\theta}{\cos\theta+\cos 2\theta+\cos 4\theta+\cos 5\theta}=\tan 3\theta.$

$$\text{L. H. S.}=\frac{(\sin 5\theta+\sin\theta)+(\sin 4\theta+\sin 2\theta)}{(\cos 5\theta+\cos\theta)+(\cos 4\theta+\cos 2\theta)}$$

$$=\frac{2\sin 3\theta\cos 2\theta+2\sin 3\theta\cos\theta}{2\cos 3\theta\cos 2\theta+2\cos 3\theta\cos\theta}$$

$$=\frac{2\sin \underset{3}{2}\theta\ (\cos 2\theta+\cos\theta)}{2\cos 3\theta\ (\cos 2\theta+\cos\theta)}=\frac{\sin 3\theta}{\cos 3\theta}=\tan 3\theta.$$

Ex. 7. Express

$\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)-\sin(A+B+C)$ as the product of three sines.

Transforming the first two terms and the last two terms into products we have

the given exp. $=2\sin C\cos(B-A)+2\cos(A+B)\sin(-C)$

$=2\sin C\cos(B-A)-2\cos(A+B)\sin C$

$=2\sin C\{\cos(B-A)-\cos(A+B)\}$

$=2\sin C\,(2\sin B\sin A)=4\sin A\sin B\sin C.$

✓**Ex. 8.** Express $4\cos\alpha\cos\beta\cos\gamma$ as the sum of four cosines.

The given exp. $=2\cos\alpha.2\cos\beta\cos\gamma$

$=2\cos\alpha\{\cos(\beta+\gamma)+\cos(\beta-\gamma)\}$

$=2\cos\alpha\cos(\beta+\gamma)+2\cos\alpha\cos(\beta-\gamma)$

$=\cos(\alpha+\beta+\gamma)+\cos(\alpha-\beta-\gamma)+\cos(\alpha+\beta-\gamma)$
$+\cos(\alpha-\beta+\gamma)$

$=\cos(\alpha+\beta+\gamma)+\cos(\beta+\gamma-\alpha)+\cos(\alpha+\beta-\gamma)$
$+\cos(\alpha-\beta+\gamma).$

Ex. 9. If $\sin A=m\sin B$, prove that

$$\tan\frac{A-B}{2}=\frac{m-1}{m+1}\tan\frac{A+B}{2}.$$

$\because\ \sin A=m\sin B,\quad\therefore\ \dfrac{\sin A}{\sin B}=\dfrac{m}{1},$

$$\therefore \text{ by Comp. \& Div. } \frac{\sin A-\sin B}{\sin A+\sin B}=\frac{m-1}{m+1},$$

$$\therefore \frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}=\frac{m-1}{m+1}, \text{ or, } \cot\frac{A+B}{2}\tan\frac{A-B}{2}=\frac{m-1}{m+1},$$

$$\text{or, } \tan\frac{A-B}{2}=\frac{m-1}{m+1}\times\frac{1}{\cot\frac{A+B}{2}}=\frac{m-1}{m+1}\tan\frac{A+B}{2}.$$

Ex. 10. Prove that $\left(\dfrac{\cos A+\cos B}{\sin A-\sin B}\right)^n+\left(\dfrac{\sin A+\sin B}{\cos A-\cos B}\right)^n$

$=2\cot^n\dfrac{A-B}{2}$ or zero, according as n is even or odd. [P.U. '33]

Here, the L. H. S.

$$=\left\{\frac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}\right\}^n+\left\{\frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{A+B}{2}\sin\frac{B-A}{2}}\right\}^n$$

$$=\left(\cot\frac{A-B}{2}\right)^n+\left\{\frac{\cos\frac{A-B}{2}}{\sin\left(-\frac{A-B}{2}\right)}\right\}^n$$

$$=\left(\cot\frac{A-B}{2}\right)^n+\left\{\frac{\cos\frac{A-B}{2}}{-\sin\frac{A-B}{2}}\right\}^n$$

$$=\left(\cot\frac{A-B}{2}\right)^n+\left(-\cot\frac{A-B}{2}\right)^n.$$

When n is even, $\left(-\cot\dfrac{A-B}{2}\right)^n$ will be positive, i.e., it will be $\left(\cot\dfrac{A-B}{2}\right)^n$, and then the given exp. $=2\cot^n\dfrac{A-B}{2}$.

Again, if n is odd, $\left(-\cot\dfrac{A-B}{2}\right)^n$ will be negative, i.e., it will be $-\left(\cot\dfrac{A-B}{2}\right)^n$, and then the given exp. $=0$.

✓**Ex. 11.** If $\sin\theta+\sin\phi=a$ and $\cos\theta+\cos\phi=b$, show that $\tan\dfrac{\theta-\phi}{2}=\pm\sqrt{\dfrac{4-a^2-b^2}{a^2+b^2}}$. [B. U. E. '63]

$$\because\ \sin\theta+\sin\phi=a,\quad\therefore\ 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}=a\ \cdots(1)$$

Again, $\because\ \cos\theta+\cos\phi=b,\quad\therefore\ 2\cos\dfrac{\theta+\phi}{2}\cos\dfrac{\theta-\phi}{2}=b\ \cdots(2)$

Now, taking the sum of the squares of (1) and (2) we have

$$4\sin^2\frac{\theta+\phi}{2}\cos^2\frac{\theta-\phi}{2}+4\cos^2\frac{\theta+\phi}{2}\cos^2\frac{\theta-\phi}{2}=a^2+b^2,$$

or, $4\cos^2\dfrac{\theta-\phi}{2}\left\{\sin^2\dfrac{\theta+\phi}{2}+\cos^2\dfrac{\theta+\phi}{2}\right\}=a^2+b^2$,

or, $4\cos^2\dfrac{\theta-\phi}{2}=a^2+b^2$ [$\because$ the other factor on the L. H. S. $=1$.]

$$\therefore\ \cos^2\frac{\theta-\phi}{2}=\frac{a^2+b^2}{4}\ \cdots\cdots(3)$$

$$\therefore\ \sin^2\frac{\theta-\phi}{2}=1-\cos^2\frac{\theta-\phi}{2}$$

$$=1-\frac{a^2+b^2}{4}=\frac{4-a^2-b^2}{4}\cdots\cdots(4)$$

Now, dividing (4) by (3) we have

$$\tan^2\frac{\theta-\phi}{2}=\frac{4-a^2-b^2}{4}\times\frac{4}{a^2+b^2}=\frac{4-a^2-b^2}{a^2+b^2}$$

$$\therefore\ \tan\frac{\theta-\phi}{2}=\pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}.$$

Ex. 12. If $\dfrac{x}{\tan(\theta+\alpha)}=\dfrac{y}{\tan(\theta+\beta)}=\dfrac{z}{\tan(\theta+\gamma)}$, prove that

$$\frac{x+y}{x-y}\sin^2(\alpha-\beta)+\frac{y+z}{y-z}\sin^2(\beta-\gamma)+\frac{z+x}{z-x}\sin^2(\gamma-\alpha)=0.$$

[Pat. '45]

From the given condition we have

$$\frac{x}{y}=\frac{\tan(\theta+\alpha)}{\tan(\theta+\beta)}\cdots\cdots(1)$$

$$\frac{y}{z}=\frac{\tan(\theta+\beta)}{\tan(\theta+\gamma)}\cdots\cdots(2)$$

$$\text{and } \frac{z}{x}=\frac{\tan(\theta+\gamma)}{\tan(\theta+\alpha)}\cdots\cdots(3)$$

Now, from (1) we get by comp. & div.

$$\frac{x+y}{x-y}=\frac{\tan(\theta+\alpha)+\tan(\theta+\beta)}{\tan(\theta+\alpha)-\tan(\theta+\beta)}=\frac{\dfrac{\sin(\theta+\alpha)}{\cos(\theta+\alpha)}+\dfrac{\sin(\theta+\beta)}{\cos(\theta+\beta)}}{\dfrac{\sin(\theta+\alpha)}{\cos(\theta+\alpha)}-\dfrac{\sin(\theta+\beta)}{\cos(\theta+\beta)}}$$

$$=\frac{\sin(\theta+\alpha)\cos(\theta+\beta)+\cos(\theta+\alpha)\sin(\theta+\beta)}{\sin(\theta+\alpha)\cos(\theta+\beta)-\cos(\theta+\alpha)\sin(\theta+\beta)}$$

$$=\frac{\sin\{(\theta+\alpha)+(\theta+\beta)\}}{\sin\{(\theta+\alpha)-(\theta+\beta)\}}=\frac{\sin(2\theta+\alpha+\beta)}{\sin(\alpha-\beta)}$$

$$=\frac{\sin(2\theta+\alpha+\beta).\sin(\alpha-\beta)}{\sin^2(\alpha-\beta)}$$ [Multiplying the numerator and the denominator by $\sin(\alpha-\beta)$]

$$\therefore\quad \frac{x+y}{x-y}.\sin^2(\alpha-\beta)=\sin(2\theta+\alpha+\beta).\sin(\alpha-\beta)$$

$$=\tfrac{1}{2}\{\cos(2\theta+2\beta)-\cos(2\theta+2\alpha)\}\cdots(4).$$

Similarly from (2) we have

$$\frac{y+z}{y-z}\sin^2(\beta-\gamma)=\tfrac{1}{2}\{\cos(2\theta+2\gamma)-\cos(2\theta+2\beta)\}\cdots(5)$$

and from (3) we have $\dfrac{z+x}{z-x}\sin^2(\gamma-\alpha)$

$$=\tfrac{1}{2}\{\cos(2\theta+2\alpha)-\cos(2\theta+2\gamma)\}\cdots(6)$$

Now, adding (4), (5), (6) we have the L. H. S. $=\frac{1}{2}\times\{0\}=0.$

Exercise 3

Express the following as a sum or difference :

1. $2\sin 5\theta \cos 2\theta$

2. $2\cos 7\theta \sin 8\theta$

3. $\cos\frac{5\theta}{2}\cos\frac{7\theta}{2}$

4. $2\sin 3A \sin (A+B)$

Express in the form of a product :—

5. $\sin 80^\circ+\sin 40^\circ$

6. $\sin 3\theta-\sin 7\theta$

7. $\cos 7A+\cos 3A$

8. $\cos 7\theta-\cos 9\theta$

Find the value of :—

9. $\cos 20^\circ+\cos 100^\circ+\cos 140^\circ$

10. $\sin 78^\circ-\sin 18^\circ+\cos 132^\circ+\cos\frac{\pi}{3}$

11. $\cos 20^\circ \cos 40^\circ \cos 80^\circ$

12. $\sqrt{3}\sin 20^\circ \sin 40^\circ \sin 80^\circ$

13. $\dfrac{\sin 75^\circ-\sin 15^\circ}{\cos 75^\circ+\cos 15^\circ}$

Prove the following identities :

14. $\dfrac{\sin\theta+\sin\alpha}{\cos\theta+\cos\alpha}=\tan\dfrac{(\theta+\alpha)}{2}$

15. $\dfrac{\sin 2\theta+\sin 3\theta}{\cos 2\theta-\cos 3\theta}=\cot\dfrac{\theta}{2}$

16. $\cos\left(\frac{\pi}{6}-\theta\right)+\cos\left(\frac{\pi}{6}+\theta\right)=\sqrt{3}\cos\theta$

17. $\sin\theta\sin\left(\frac{\pi}{3}-\theta\right)\sin\left(\frac{\pi}{3}+\theta\right)=\frac{1}{4}\sin 3\theta$

18. $\dfrac{\cos(2A-3B)+\cos 3B}{\sin(2A-3B)+\sin 3B}=\cot A.$

✓**19.** $\cos 95^\circ+\cos 25^\circ=\dfrac{1}{\sqrt{2}}(\cos 10^\circ+\sin 10^\circ)$

20. $\cos\theta+\cos\left(\frac{2\pi}{3}+\theta\right)+\cos\left(\frac{2\pi}{3}-\theta\right)=0$

✓**21.** $\frac{\cos 10^\circ-\sin 10^\circ}{\cos 10^\circ+\sin 10^\circ}=\tan 35^\circ$

22. $\sin^2 5\theta-\sin^2 3\theta=\sin 8\theta\sin 2\theta$

23. $\frac{\cos(A+B)-2\cos A+\cos(A-B)}{\sin(A+B)-2\sin A+\sin(A-B)}=\cot A.$

24. $\cos 10^\circ-\sin 40^\circ=\sin 20^\circ$

25. $\frac{\sin 7A-\sin 3A-\sin 5A+\sin A}{\cos 7A+\cos 3A-\cos 5A-\cos A}=\tan 2A$

✓**26.** $\sin^2\theta+\sin^2\left(\frac{2\pi}{3}+\theta\right)+\sin^2\left(\frac{2\pi}{3}-\theta\right)=\frac{3}{2}$

27. $4\cos\theta\cos(120^\circ+\theta)\cos(120^\circ-\theta)=\cos 3\theta$

✓**28.** $\cos A+\cos B+\cos C+\cos(A+B+C)$

$=4\cos\frac{B+C}{2}\cos\frac{C+A}{2}\cos\frac{A+B}{2}.$ [A. U. '45]

29. Express $\sin 2A+\sin 2B+\sin 2C-\sin 2(A+B+C)$ as the product of three sines.

30. Express $4\sin A\cos B\cos C$ as the sum of four sines.

✓**31.** If $\operatorname{cosec} A+\sec A=\operatorname{cosec} B+\sec B$, then $\tan A\tan B=\cot\frac{1}{2}(A+B)$. [P. U. '36]

32. Express $\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)$ $-\sin(A+B+C)$ as the product of three sines.

✓**33.** If $\cos A=k\cos B$, shew that

$$\cot\tfrac{1}{2}(A+B)=\frac{k+1}{k-1}\tan\tfrac{1}{2}(B-A).$$

34. If $\cos A+\cos B=\frac{1}{3}$ and $\sin A+\sin B=\frac{1}{4}$, find the value of $\tan\frac{1}{2}(A+B)$.

35. Simplify $\frac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A}$.

[A. U. '48]

36. If $\sin A+\sin B=a$, and $\cos A+\cos B=b$, find the value of $\cot\dfrac{A-B}{2}$.

✓**37.** If $x\cos\alpha+y\sin\alpha=k=x\cos\beta+y\sin\beta$, prove that

$$\frac{x}{\cos\frac{1}{2}(\alpha+\beta)}=\frac{y}{\sin\frac{1}{2}(\alpha+\beta)}=\frac{k}{\cos\frac{1}{2}(\alpha-\beta)}.$$

MULTIPLE ANGLES

19. Trigonometrical ratios of angle 2A.

Angles 2A, 3A, 4A, etc. are called multiple angles, as 2A, 3A, etc. are multiples of A.

(*a*) It has been proved before that for all values of A and B,

$$\sin(A+B)=\sin A\cos B+\cos A\sin B, \text{ and}$$
$$\cos(A+B)=\cos A\cos B-\sin A\sin B.$$

Now putting $B=A$ in the first formula, we have

$$\mathbf{\sin 2A}=\sin A\cos A+\cos A\sin A=\mathbf{2\sin A\cos A}\cdots(1)$$

Putting $B=A$ in the second formula we have

$$\mathbf{\cos 2A}=\cos A.\cos A-\sin A.\sin A=\mathbf{\cos^2 A-\sin^2 A}\cdots(2)$$
$$=\cos^2 A-(1-\cos^2 A)=\mathbf{2\cos^2 A-1}\cdots(3)$$

[Or, from (2)] $=(1-\sin^2 A)-\sin^2 A=\mathbf{1-2\sin^2 A}\cdots(4)$

Hence $\mathbf{\cos 2A=\cos^2 A-\sin^2 A=2\cos^2 A-1=1-2\sin^2 A}$.

Corollary : From (3) and (4) we have by transposition

$\mathbf{1+\cos 2A=2\cos^2 A}\cdots(5)$ and $\mathbf{1-\cos 2A=2\sin^2 A}\cdots(6)$

$$\therefore\quad \frac{1-\cos 2A}{1+\cos 2A}=\frac{2\sin^2 A}{2\cos^2 A}=\tan^2 A.$$

(*b*) It has been proved before that

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

and $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot B+\cot A}$.

Putting $B = A$ in the first formula we get

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \cdots (7)$$

Putting $B = A$ in the second formula we get

$$\cot 2A = \frac{\cot^2 A - 1}{2 \cot A} \cdots (8)$$

[**N. B.** As the addition formulæ have been proved for all values of A and B, the above formulas derived from them are also true for all values of A.]

20. Trigonometrical ratios of angle 3A.

From the formula we get

(i) $$\begin{aligned} \sin 3A &= \sin (2A + A) = \sin 2A \cos A + \cos 2A \sin A \\ &= 2 \sin A \cos A . \cos A + (1 - 2 \sin^2 A) . \sin A \\ &= 2 \sin A \cos^2 A + \sin A - 2 \sin^3 A \\ &= 2 \sin A (1 - \sin^2 A) + \sin A - 2 \sin^3 A \\ &= 2 \sin A - 2 \sin^3 A + \sin A - 2 \sin^3 A \\ &= 3 \sin A - 4 \sin^3 A. \end{aligned}$$

$$\therefore \quad \sin 3A = 3 \sin A - 4 \sin^3 A.$$

(ii) $$\begin{aligned} \cos 3A &= \cos (2A + A) = \cos 2A \cos A - \sin 2A \sin A \\ &= (2 \cos^2 A - 1) \cos A - 2 \sin A \cos A . \sin A \\ &= 2 \cos^3 A - \cos A - 2 \sin^2 A \cos A \\ &= 2 \cos^3 A - \cos A - 2 \cos A (1 - \cos^2 A) \\ &= 4 \cos^3 A - 3 \cos A. \end{aligned}$$

$$\therefore \quad \cos 3A = 4 \cos^3 A - 3 \cos A.$$

[**N. B.** We notice that in finding sin 3A and cos 3A above we have to put the value of cos 2A. But cos 2A has two values, viz., $2 \cos^2 A - 1$ and $1 - 2 \sin^2 A$. It is to be noted that the value of cos 2A must be written in terms of sine in finding sin 3A and in terms of cosine in finding cos 3A.]

(iii) $\mathbf{\tan 3A} = \tan(2A+A) = \dfrac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$ [C. Pre-U. '63]

$$= \frac{\dfrac{2\tan A}{1-\tan^2 A} + \tan A}{1 - \dfrac{2\tan A}{1-\tan^2 A}.\tan A} = \frac{\dfrac{2\tan A + \tan A - \tan^3 A}{1-\tan^2 A}}{\dfrac{1-\tan^2 A - 2\tan^2 A}{1-\tan^2 A}}$$

$$= \mathbf{\frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}}.$$

(iv) To express **sin 2**A and **cos 2**A in terms of **tan** A.

[C. U. '31]

$$\sin 2A = 2\sin A\cos A = \frac{2\sin A\cos A}{\sin^2 A + \cos^2 A}$$

[$\because \quad \sin^2 A + \cos^2 A = 1$]

$$= \frac{\dfrac{2\sin A\cos A}{\cos^2 A}}{\dfrac{\sin^2 A}{\cos^2 A} + \dfrac{\cos^2 A}{\cos^2 A}}$$ [dividing the numerator and the denominator by $\cos^2 A$.]

$$= \frac{2\tan A}{1+\tan^2 A}.$$

[**Alt. proof :** $\sin 2A = 2\sin A\cos A$

$$= 2\frac{\sin A}{\cos A}.\cos^2 A = 2\tan A.\cos^2 A$$

$$= 2\tan A\,\frac{1}{\sec^2 A} = \frac{2\tan A}{1+\tan^2 A}.$$

$$\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A - \cos^2 A.\frac{\sin^2 A}{\cos^2 A}$$

$$= \cos^2 A\,(1 - \tan^2 A) = \frac{1-\tan^2 A}{\sec^2 A} = \frac{1-\tan^2 A}{1+\tan^2 A}.$$

[**N. B.** By a process similar to that shown above the trigonometrical ratios of any multiple of angle A can be expressed in terms of those of angle A.]

Example (4)

Ex. 1. Express $\cos 4\theta$ in terms of $\sin\theta$.

$$\cos 4\theta=\cos 2\,(2\theta)=1-2\sin^2 2\theta=1-2\,(\sin 2\theta)^2$$
$$=1-2(2\sin\theta\cos\theta)^2=1-8\sin^2\theta\cos^2\theta$$
$$=1-8\sin^2\theta(1-\sin^2\theta)=1-8\sin^2\theta+8\sin^4\theta.$$

Ex. 2. If $\sin A=\frac{3}{5}$, find the value of $\cos 2A$.

$$\cos 2A=1-2\sin^2 A=1-2\times\tfrac{9}{25}=\tfrac{7}{25}.$$

✓**Ex. 3.** Find $\tan A$, when $\cos 2A=\frac{24}{25}$.

$$\because\quad \cos 2A=\frac{1-\tan^2 A}{1+\tan^2 A},\quad \therefore\quad \frac{1-\tan^2 A}{1+\tan^2 A}=\frac{24}{25},$$

$$\therefore\quad \frac{2}{2\tan^2 A}=\frac{49}{1}\ \text{(by Comp. \& Div.)}.$$

or, $\tan^2 A=\frac{1}{49}$, $\therefore\ \tan A=\frac{1}{7}$.

Ex. 4. If $\cos A=\frac{4}{5}$, find the value of $\sin 3A$.

$$\because\quad \cos A=\tfrac{4}{5},\quad \therefore\quad \sin A=\sqrt{1-\cos^2 A}=\sqrt{1-\tfrac{16}{25}}$$
$$=\sqrt{\tfrac{9}{25}}=\tfrac{3}{5}.$$

Now, $\sin 3A=3\sin A-4\sin^3 A=3\times\frac{3}{5}-4.\left(\frac{3}{5}\right)^3$

$$=\tfrac{9}{5}-\tfrac{108}{125}=\tfrac{117}{125}.$$

Ex. 5. Prove that $\cos^2(45^\circ-\theta)-\sin^2(45^\circ-\theta)=\sin 2\theta$.

L. H. S. $=\cos^2 A-\sin^2 A$ [putting A for $45^\circ-\theta$]

$$=\cos 2A=\cos 2\,(45^\circ-\theta)=\cos(90^\circ-2\theta)=\sin 2\theta.$$

Ex. 6. Show that $\sin 8\theta=8\sin\theta\cos\theta\cos 2\theta\cos 4\theta$.

$$\sin 8\theta=\sin 2\,(4\theta)=2\sin 4\theta\cos 4\theta$$
$$=2.2\sin 2\theta\cos 2\theta.\cos 4\theta$$

[$\because\ \sin 4\theta=2\sin 2\theta\cos 2\theta$]

$$=2.2.2\sin\theta\cos\theta\cos 2\theta\cos 4\theta$$

[$\because\ \sin 2\theta=2\sin\theta\cos\theta$]

$$=8\sin\theta\cos\theta\cos 2\theta\cos 4\theta.$$

✓**Ex. 7.** Prove that $\cos^6 A+\sin^6 A=1-3\sin^2 A\cos^2 A$
$=\frac{1}{4}(1+3\cos^2 2A)$.

$$\begin{aligned}\text{L. H. S.}&=(\cos^2 A)^3+(\sin^2 A)^3\\&=(\cos^2 A+\sin^2 A)(\cos^4 A-\cos^2 A\sin^2 A+\sin^4 A)\\&=\cos^4 A-\sin^2 A\cos^2 A+\sin^4 A\\&\qquad\qquad[\because\ \cos^2 A+\sin^2 A=1]\\&=(\cos^2 A+\sin^2 A)^2-3\sin^2 A\cos^2 A\\&=\mathbf{1-3\sin^2 A\cos^2 A}.\quad[\text{ proved }]\\&=1-3\times(\tfrac{1}{2}\sin 2A)^2\quad[\because\ \sin 2A=2\sin A\cos A]\\&=1-\tfrac{3}{4}\sin^2 2A=1-\tfrac{3}{4}(1-\cos^2 2A)=\tfrac{1}{4}+\tfrac{3}{4}\cos^2 2A\\&=\tfrac{1}{4}(\mathbf{1+3\cos^2 2A}).\quad[\text{ proved }]\end{aligned}$$

✓**Ex. 8.** Show that $\cot\theta-\tan\theta=2\cot 2\theta$.

$$\text{L. H. S.}=\cot\theta-\frac{1}{\cot\theta}=\frac{\cot^2\theta-1}{\cot\theta}=2.\frac{\cot^2\theta-1}{2\cot\theta}=2\cot 2\theta.$$

Ex. 9. Prove that $\dfrac{1-\cos 2\theta+\sin 2\theta}{1+\cos 2\theta+\sin 2\theta}=\tan\theta$. [C.U. '38]

$$\text{L. H. S.}=\frac{(1-\cos 2\theta)+\sin 2\theta}{(1+\cos 2\theta)+\sin 2\theta}$$

$$=\frac{2\sin^2\theta+\sin 2\theta}{2\cos^2\theta+\sin 2\theta}\quad[\text{ Art. 19, formulas 5 \& 6 }]$$

$$=\frac{2\sin^2\theta+2\sin\theta\cos\theta}{2\cos^2\theta+2\sin\theta\cos\theta}=\frac{2\sin\theta\,(\sin\theta+\cos\theta)}{2\cos\theta\,(\cos\theta+\sin\theta)}=\tan\theta.$$

✓**Ex. 10.** If $\tan\theta=\dfrac{x}{y}$, find the value of $x\sin 2\theta+y\cos 2\theta$.

[B. H. U. '40]

$$\because\ \tan\theta=\frac{x}{y},\quad\therefore\ \frac{\sin\theta}{\cos\theta}=\frac{x}{y},\quad\therefore\ y\sin\theta=x\cos\theta.$$

$$\begin{aligned}\text{Now } x\sin 2\theta+y\cos 2\theta&=2.\,x\sin\theta\cos\theta+y(1-2\sin^2\theta)\\&=2\sin\theta.\,x\cos\theta+y(1-2\sin^2\theta)\\&=2\sin\theta.\,y\sin\theta+y(1-2\sin^2\theta)\\&=2y\sin^2\theta+y(1-2\sin^2\theta)=y.\end{aligned}$$

✓**Ex. 11.** Prove that $\dfrac{\sec 8A-1}{\sec 4A-1}=\dfrac{\tan 8A}{\tan 2A}$. [B. H. U. '57]

$$\text{L. H. S.}=\frac{\dfrac{1}{\cos 8A}-1}{\dfrac{1}{\cos 4A}-1}=\frac{1-\cos 8A}{\cos 8A}\times\frac{\cos 4A}{1-\cos 4A}$$

$$=\frac{2\sin^2 4A.\cos 4A}{\cos 8A.\,2\sin^2 2A}$$

$$=\frac{2\sin 4A\cos 4A\times\sin 4A}{\cos 8A\,.\,2\sin^2 2A}=\frac{\sin 8A}{\cos 8A}\times\frac{\sin 4A}{2\sin^2 2A}$$

$$=\tan\ 8A\times\frac{2\sin 2A\cos 2A}{2\sin^2 2A}=\tan 8A\times\frac{\cos 2A}{\sin 2A}$$

$$=\tan 8A\cot 2A=\tan 8A\times\frac{1}{\tan 2A}=\frac{\tan 8A}{\tan 2A}.$$

✓**Ex. 12.** Prove that $\tan 3A-\tan 2A-\tan A$

$=\tan 3A\tan 2A\tan A.$

$$\text{L. H. S.}=\tan(2A+A)-\tan 2A-\tan A$$

$$=\frac{\tan 2A+\tan A}{1-\tan 2A\tan A}-\tan 2A-\tan A$$

$$=\frac{\tan 2A+\tan A-\tan 2A-\tan A+(\tan 2A+\tan A)\tan 2A\tan A}{1-\tan 2A\tan A}$$

$$=\frac{\tan 2A+\tan A}{1-\tan 2A\tan A}.\tan 2A\tan A=\tan 3A\tan 2A\tan A.$$

✓**Ex. 13.** Show that $\cos^2\theta+\cos^2(\alpha+\theta)$

$-2\cos\alpha\cos\theta\cos(\alpha+\theta)$ is independent of θ. [P. U. '46]

The given exp. $=\cos^2\theta+(\cos\alpha\cos\theta-\sin\alpha\sin\theta)^2$

$$-2\cos\alpha\cos\theta\,(\cos\alpha\cos\theta-\sin\theta\sin\alpha)$$

$$=\cos^2\theta+\cos^2\alpha\cos^2\theta+\sin^2\alpha\sin^2\theta-2\cos\alpha\cos\theta\sin\alpha\sin\theta$$

$$-2\,\mathrm{ccs}^2\alpha\cos^2\theta+2\cos\alpha\cos\theta\sin\alpha\sin\theta$$

$=\cos^2\theta-\cos^2\alpha\cos^2\theta+\sin^2\alpha\sin^2\theta$

$=\cos^2\theta-\cos^2\theta\,(1-\sin^2\alpha)+\sin^2\alpha\,(1-\cos^2\theta)$

$=\cos^2\theta-\cos^2\theta+\cos^2\theta\sin^2\alpha+\sin^2\alpha-\cos^2\theta\sin^2\alpha$

$=\sin^2\alpha$, which is independent of θ.

✓**Ex. 14.** Prove that $\dfrac{\tan 2^n\theta}{\tan\theta}$

$$=(1+\sec 2\theta)(1+\sec 2^2\theta)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta).$$

$$\because\quad 1+\sec 2\theta=1+\frac{1}{\cos 2\theta}=1+\frac{1}{\dfrac{1-\tan^2\theta}{1+\tan^2\theta}}=1+\frac{1+\tan^2\theta}{1-\tan^2\theta}$$

$$=\frac{1-\tan^2\theta+1+\tan^2\theta}{1-\tan^2\theta}=\frac{2}{1-\tan^2\theta},$$

$\therefore\quad \tan\theta\,(1+\sec 2\theta)=\dfrac{2\tan\theta}{1-\tan^2\theta}$ (multiplying both sides by $\tan\theta$)

$$=\tan 2\theta.$$

Similarly it can be proved that $\tan 2\theta.(1+\sec 2^2\theta)=\tan 2^2\theta$, $\tan 2^2\theta(1+\sec 2^3\theta)=\tan 2^3\theta$, ……, and $\tan 2^{n-1}\theta\,(1+\sec 2^n\theta)=\tan 2^n\theta$.

$$\therefore\quad \tan\theta(1+\sec 2\theta)(1+\sec 2^2\theta)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)=\tan 2^n\theta,$$

$$\therefore\quad \frac{\tan 2^n\theta}{\tan\theta}=(1+\sec 2\theta)(1+\sec 2^2\theta)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta).$$

Exercise (4)

1. If $\cot A=3$, find $\tan 2A$.

2. If $\sec\theta=3$, find the value of $\cos 3\theta$.

3. If $\sin 2A=\cdot 28$, find $\tan A$.

4. (i) Simplify $\sin 3A\operatorname{cosec} A-\dfrac{\cos 3A}{\cos A}$.

(ii) Express $\sin 2A$ and $\cos 2A$ in terms of $\tan A$.

[C. U. '31]

Prove the following indentities :—

5. $\dfrac{\sin 2\theta}{1-\cos 2\theta}=\cot\theta.$

6. $\cot 2A+\tan A=\operatorname{cosec} 2A$ [C. U. '47]

7. $\cos^4 A-\sin^4 A=\cos 2A.$ 8. $\dfrac{1+\tan^2 A}{1-\tan^2 A}=\sec 2A.$

9. $\dfrac{\cos 2\theta}{1-\sin 2\theta}=\cot\left(\dfrac{\pi}{4}-\theta\right)$

10. $\cot A+\cot(60^\circ+A)+\cot(120^\circ+A)=3\cot 3A.$

[Pat. '45]

11. $\dfrac{\cos 3\theta+\sin 3\theta}{\cos\theta-\sin\theta}=1+2\sin 2\theta.$

12. $\sin^3 A+\sin^3(120^\circ+A)+\sin^3(240^\circ+A)=-\frac{3}{4}\sin 3A.$

[P. U. '39]

13. $\tan\left(\dfrac{\pi}{4}+\theta\right)-\tan\left(\dfrac{\pi}{4}-\theta\right)=2\tan 2\theta.$

14. $\cos^3 A\cos 3A+\sin^3 A\sin 3A=\cos^3 2A.$ [P. U. '39, '43]

15. $\dfrac{\sin A+\sin 2A+\sin 4A+\sin 5A}{\cos A+\cos 2A+\cos 4A+\cos 5A}=\tan 3A.$

16. $\cos^2 A+\cos^2\left(A+\dfrac{\pi}{3}\right)+\cos^2\left(A-\dfrac{\pi}{3}\right)=\dfrac{3}{2}.$ [C. U. '43]

17. $\cos 3\theta\sin 2\theta-\cos 4\theta\sin\theta=\dfrac{\cos 2\theta}{\operatorname{cosec}\theta}.$

[B. H. U. '48]

18. $2\operatorname{cosec} 2A=\tan A+\cot A.$

19. $4\sin^3\theta\cos 3\theta+4\cos^3\theta\sin 3\theta=3\sin 4\theta.$

[*Hints :* $4\sin^3\theta=3\sin\theta-\sin 3\theta$, and
$4\cos^3\theta=3\cos\theta+\cos 3\theta$]

20. $\dfrac{\sin(A+3B)+\sin(3A+B)}{\sin 2A+\sin 2B}=2\cos(A+B).$ [A. U. '47]

21. $\cos 4x-\cos 4y=8(\cos x-\cos y)(\cos x+\cos y)\times$
$(\cos x-\sin y)(\cos x+\sin y).$ [P. U. '36]

✓22. $\dfrac{\sin 4A\,(1-\cos 2A)}{\cos 2A\,(1-\cos 4A)}=\tan A.$

23. $\dfrac{1}{\tan 3\alpha-\tan\alpha}-\dfrac{1}{\cot 3\alpha-\cot\alpha}=\cot 2\alpha.$

✓24. If $2\tan\alpha=3\tan\beta$, show that $\tan(\alpha-\beta)=\dfrac{\sin 2\beta}{5-\cos 2\beta}$.
[C. U. '46 ; P. U. '47]

25. Find the value of $\cos^2\theta+\cos^2(120^\circ-\theta)+\cos^2(120^\circ+\theta)$.

✓26. If $\tan\theta=\sec 2\alpha$, prove that $\sin 2\theta=\dfrac{1-\tan^4\alpha}{1+\tan^4\alpha}$.
[P. U. '40]

✓27. If α and β are acute angles and $\cos 2\alpha=\dfrac{3\cos 2\beta-1}{3-\cos 2\beta}$,
show that $\tan\alpha=\sqrt{2}\tan\beta$. [C. U. '41]

✓28. If $\tan\theta=\frac{1}{7}$ and $\tan\phi=\frac{1}{3}$, show that $\cos 2\theta=\sin 4\phi$.
[A. U. '50]

✓29. If $\theta=\dfrac{\pi}{2^n+1}$, prove that

$$2^n\cos\theta\cos 2\theta\cos 2^2\theta\ldots\cos 2^{n-1}\theta=1.$$

✓30. Prove that $\dfrac{2\cos 2^n\theta+1}{2\cos\theta+1}=(2\cos\theta-1)(2\cos 2\theta-1)\times$
$(2\cos 2^2\theta-1)\cdots(2\cos 2^{n-1}\theta-1).$

Submultiple Angles

The angles $\frac{A}{2}$, $\frac{A}{3}$, $\frac{A}{4}$, etc. are called Submultiple Angles. They are submultiples of angle A.

21. We have proved the following formulas in case of multiple angles.

(1) $\sin 2A=2\sin A\cos A$

(2) $\cos 2A=\cos^2A-\sin^2A=2\cos^2A-1=1-2\sin^2A$;
$[1+\cos 2A=2\cos^2A\ ;\ 1-\cos 2A=2\sin^2A\,]$

(3) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$.

These formulas will hold good whenever an angle is double of another.

Putting $\frac{A}{2}$ for A (so putting $2.\frac{A}{2}$ or A for $2A$) in the above formulas we have respectively ;

(1) $\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$

(2) $\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} = 2 \cos^2 \frac{A}{2} - 1 = 1 - 2 \sin^2 \frac{A}{2}$

$$\left[1 + \cos A = 2 \cos^2 \frac{A}{2} ; \quad 1 - \cos A = 2 \sin^2 \frac{A}{2} \right]$$

(3) $\tan A = \dfrac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$.

22. The following relations regarding multiple angles have been proved before :

(4) $\sin 3A = 3 \sin A - 4 \sin^3 A$

(5) $\cos 3A = 4 \cos^3 A - 3 \cos A$

(6) $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$.

These formulas are true whenever an angle is thrice another angle.

Now, putting $\frac{A}{3}$ for A (so $3.\frac{A}{3}$ or A for $3A$) in the above formulas we respectively have :

(4) $\sin A = 3 \sin \frac{A}{3} - 4 \sin^3 \frac{A}{3}$

(5) $\cos A = 4 \cos^3 \frac{A}{3} - 3 \cos \frac{A}{3}$

(6) $$\tan A=\frac{3\tan\frac{A}{3}-\tan^3\frac{A}{3}}{1-3\tan^2\frac{A}{3}}.$$

Corollary : (i) $\because \sin 2A=\dfrac{2\tan A}{1+\tan^2 A}$,

$\therefore$ putting $\frac{A}{2}$ for A we get $\sin A=\dfrac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

(ii) $\because \cos 2A=\dfrac{1-\tan^2 A}{1+\tan^2 A}$, $\therefore \cos A=\dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$.

23. To find the trigonometrical ratios of $\frac{A}{2}$ in terms of cos A.

$\because 1-\cos A=2\sin^2\frac{A}{2}$, $\therefore \sin^2\frac{A}{2}=\frac{1}{2}(1-\cos A)$

$\therefore \sin\frac{A}{2}=\pm\sqrt{\frac{1-\cos A}{2}}$......(i)

Again, $\because 1+\cos A=2\cos^2\frac{A}{2}$, $\therefore \cos^2\frac{A}{2}=\frac{1}{2}(1+\cos A)$,

$\therefore \cos\frac{A}{2}=\pm\sqrt{\frac{1}{2}(1+\cos A)}$......(ii)

Hence, $\tan\frac{A}{2}=\dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\pm\sqrt{\dfrac{1-\cos A}{1+\cos A}}$......(iii)

[The other ratios can now be easily deduced]

24. Ambiguity of signs.

In the above article we find that $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ have two values, equal but opposite in sign, derived from the value of cos A. The reason for this ambiguity is that when only cos A is known and not A, then A and $\frac{A}{2}$ may have a series of values. If α be the least of the series of values, then $A=2n\pi\pm\alpha$ (where n is any integer).

So, we are really finding the values of $\sin\frac{1}{2}(2n\pi\pm\alpha)$ and $\cos\frac{1}{2}(2n\pi\pm\alpha)$ in finding the values of $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$.

$$\text{Now,}\quad \sin\tfrac{1}{2}(2n\pi\pm\alpha)=\sin\left(n\pi\pm\frac{\alpha}{2}\right)$$

$$=\sin n\pi\cos\frac{\alpha}{2}\pm\cos n\pi\sin\frac{\alpha}{2}=\pm\sin\frac{\alpha}{2}$$

$$[\because\ \sin n\pi=0 \text{ and } \cos n\pi=\pm1]$$

$$\text{Again,}\quad \cos\frac{1}{2}(2n\pi\pm\alpha)=\cos\left(n\pi\pm\frac{\alpha}{2}\right)$$

$$=\cos n\pi\cos\frac{\alpha}{2}\mp\sin n\pi\sin\frac{\alpha}{2}=\pm\cos\frac{\alpha}{2}.$$

Hence, if only cos A is given and not A, we have two values of $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$.

If, however, the value of A also or the final position of the boundary line of the angle A is known, there will be no ambiguity regarding the sign of the value of $\sin\frac{A}{2}$ or $\cos\frac{A}{2}$, for in such cases the quadrant in which $\frac{A}{2}$ lies and consequently the sign of $\sin\frac{A}{2}$ or $\cos\frac{A}{2}$ will be definitely known. Hence the sign to be used in the above formulas depends on the quadrant in which $\frac{A}{2}$ lies.

Example. If $\cos 330^\circ = \frac{\sqrt{3}}{2}$, find the value of $\sin 165^\circ$ and $\cos 165^\circ$.

$$\sin 165^\circ = \pm\sqrt{\frac{1-\cos 330^\circ}{2}} = \pm\sqrt{\tfrac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right)}$$

$$= \pm\sqrt{\tfrac{1}{2}\left(\frac{2-\sqrt{3}}{2}\right)}$$

$$= \pm\sqrt{\frac{1}{2}\times\frac{4-2\sqrt{3}}{4}} = \pm\sqrt{\frac{1}{2}\times\left(\frac{\sqrt{3}-1}{2}\right)^2} = \pm\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Again, $\cos 165^\circ = \sqrt{\tfrac{1}{2}(1+\cos 330^\circ)} = \pm\sqrt{\frac{1}{2}\times\left(\frac{\sqrt{3}+1}{2}\right)^2}$

$$= \pm\frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Now, 165° being in the second (quadrant) its sine is positive and cosine negative.

$$\therefore \quad \sin 165^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}, \text{ and } \cos 165^\circ = -\frac{\sqrt{3}+1}{2\sqrt{2}}.$$

25. To find the value of $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ in terms of $\sin A$.

$$\because \quad \sin^2\frac{A}{2}+\cos^2\frac{A}{2} = 1 \cdots\cdots(1)$$

$$\text{and } 2\sin\frac{A}{2}\cos\frac{A}{2} = \sin A \cdots(2),$$

$\therefore$ adding (1) and (2) we have $\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)^2 = 1+\sin A$,

$$\therefore \quad \sin\frac{A}{2}+\cos\frac{A}{2} = \pm\sqrt{1+\sin A} \cdots(3)$$

Again, subtracting (2) from (1) we get

$$\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)^2=1-\sin A,$$

$$\therefore\quad \sin\frac{A}{2}-\cos\frac{A}{2}=\pm\sqrt{1-\sin A}\cdots\cdots(4).$$

Now, adding (3) and (4) we have

$$2\sin\frac{A}{2}=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A},$$

$$\therefore\quad \sin\frac{A}{2}=\pm\tfrac{1}{2}\sqrt{1+\sin A}\pm\tfrac{1}{2}\sqrt{1-\sin A}\cdots\cdots(5)$$

Subtracting (4) from (3) we have

$$\cos\frac{A}{2}=\pm\tfrac{1}{2}\sqrt{1+\sin A}\mp\tfrac{1}{2}\sqrt{1-\sin A}\cdots\cdots(6).$$

26. Ambiguity of signs.

From the formulas (5) and (6) above, we find that for one value of $\sin A$, either $\sin\frac{A}{2}$ or $\cos\frac{A}{2}$ has four values each. If only $\sin A$ is given, but not A, then A and hence $\frac{A}{2}$ may have a series of values. Let α be the least of this series of values.

Then $A=n\pi+(-1)^n\alpha$ and $\therefore\ \sin\frac{A}{2}=\sin\frac{1}{2}\{n\pi+(-1)^n\alpha\}$ and $\cos\frac{A}{2}=\cos\frac{1}{2}\{n\pi+(-1)^n\alpha\}$.

Now, (i) if n be an even integer, let $n=2m$.

$$\therefore\quad \sin\tfrac{1}{2}\{n\pi+(-1)^n\alpha\}=\sin\tfrac{1}{2}\{2m\pi+(-1)^{2m}\alpha\}$$

$$=\sin\left(m\pi+\frac{\alpha}{2}\right)$$

$$=\sin m\pi\cos\frac{\alpha}{2}+\cos m\pi\sin\frac{\alpha}{2}$$

$$=\pm\sin\frac{\alpha}{2}\quad[\because\ \sin m\pi=0\text{ and }\cos m\pi=\pm1]$$

(ii) If n be an odd integer, let $n=2m+1$.

$$\therefore \quad \sin \tfrac{1}{2}\{n\pi+(-1)^n\alpha\}=\sin \tfrac{1}{2}\{(2m+1)\pi+(-1)^{2m+1}\alpha\}$$

$$=\sin\left(m\pi+\frac{\pi}{2}-\frac{\alpha}{2}\right) \quad [\because (-1)^{2m+1}=-1]$$

$$=\sin m\pi \cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)+\cos m\pi \sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)=\pm\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right).$$

Hence, we notice that if only $\sin A$ is known and not A, $\sin\frac{A}{2}$ will have four values.

Similarly it can be shown that in such cases $\cos\frac{A}{2}$ also will have four values, viz., $\cos\frac{A}{2}=\pm\cos\frac{\alpha}{2}$ or $\pm\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)$ for one value of A.

Now, we have $\sin\frac{A}{2}+\cos\frac{A}{2}=\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\frac{A}{2}+\frac{1}{\sqrt{2}}\cos\frac{A}{2}\right)$

$$=\sqrt{2}\sin\left(\frac{A}{2}+\frac{\pi}{4}\right)\left[\because \quad \sin\frac{\pi}{4}=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}\right]$$

$$\therefore \quad \sqrt{2}\sin\left(\frac{A}{2}+\frac{\pi}{4}\right)=\sin\frac{A}{2}+\cos\frac{A}{2}=\pm\sqrt{1+\sin A},$$

Again, $\sin\frac{A}{2}-\cos\frac{A}{2}=\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\frac{A}{2}-\frac{1}{\sqrt{2}}\cos\frac{A}{2}\right)$

$$=\sqrt{2}\sin\left(\frac{A}{2}-\frac{\pi}{4}\right),$$

$$\therefore \quad \sqrt{2}\sin\left(\frac{A}{2}-\frac{\pi}{4}\right)=\sin\frac{A}{2}-\cos\frac{A}{2}=\pm\sqrt{1-\sin A}.$$

Hence, if the value of A is known, it will be definitely known whether $\sin\left(\frac{A}{2}+\frac{\pi}{4}\right)$ and $\sin\left(\frac{A}{2}-\frac{\pi}{4}\right)$ are positive or negative and there will be no ambiguity about the signs.

Thus the sign to be used in the above formulas depends on the quadrant in which $\frac{A}{2}$ lies.

27. Value of $\tan \frac{A}{2}$ in terms of $\tan A$.

From the formula we have $\tan A = \dfrac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$,

$\therefore$ $\tan A - \tan A \tan^2 \frac{A}{2} = 2 \tan \frac{A}{2}$, (by cross multiplication)

or, $\tan A \tan^2 \frac{A}{2} + 2 \tan \frac{A}{2} - \tan A = 0$, it is a quadratic equation. By solving it we have

$$\tan \frac{A}{2} = \frac{-2 \pm \sqrt{(2)^2 - 4 \tan A \times - \tan A}}{2 \tan A}$$

$$= \frac{-2 \pm \sqrt{4 + 4 \tan^2 A}}{2 \tan A} = \frac{-2 \pm 2\sqrt{1 + \tan^2 A}}{2 \tan A}$$

$$= \frac{-1 \pm \sqrt{1 + \tan^2 A}}{\tan A}.$$

[Here also the reason for the ambiguity in signs is similar to the reasons of the previous two cases.]

28. Ratios of angles of 18°, 36°, 54°, 72°.

(i) Let $A = 18^\circ$, then $5A = 90^\circ$, $\therefore$ $2A = 90^\circ - 3A$.

$\therefore$ $\sin 2A = \sin (90^\circ - 3A) = \cos 3A = 4 \cos^3 A - 3 \cos A$,

or, $2 \sin A \cos A = \cos A \, (4 \cos^2 A - 3)$.

Now, $\because$ $A = 18^\circ$, $\therefore$ $\cos A \neq 0$,

$\therefore$ $2 \sin A = 4 \cos^2 A - 3 = 4(1 - \sin^2 A) - 3 = 1 - 4 \sin^2 A$,

or, $4 \sin^2 A + 2 \sin A - 1 = 0$,

$$\therefore \quad \sin A = \frac{-2 \pm \sqrt{2^2 - 4 \times 4 \times -1}}{2 \times 4} = \frac{-2 \pm \sqrt{4 + 16}}{8}$$

$$= \frac{-2 \pm 2\sqrt{5}}{8} = \frac{\pm \sqrt{5} - 1}{4}.$$

Since A is an acute angle here, sin A must be positive, so the negative value of A is inadmissible.

$$\therefore \quad \mathbf{\sin 18^\circ = \tfrac{1}{4}(\sqrt{5}-1).}$$

Again, $\mathbf{\cos 18^\circ} = +\sqrt{1-\sin^2 18^\circ} = \sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2}$

$$= \sqrt{\frac{10+2\sqrt{5}}{16}} = \tfrac{1}{4}\sqrt{\mathbf{10+2\sqrt{5}}}.$$

(ii) $\mathbf{\cos 36^\circ} = \cos 2.18^\circ = 1-2\sin^2 18^\circ$

$$= 1-2\times\frac{6-2\sqrt{5}}{16} = \tfrac{1}{4}(\sqrt{5}+1)$$

$$\mathbf{\sin 36^\circ} = \sqrt{1-\cos^2 36^\circ} = \sqrt{1-\tfrac{1}{16}(\sqrt{5}+1)^2}$$

$$\sqrt{1-\tfrac{1}{16}(6+2\sqrt{5})} = \tfrac{1}{4}\sqrt{\mathbf{10-2\sqrt{5}}}.$$

(iii) $\because$ angle 54° is the complement of angle 36°,

$\therefore$ $\mathbf{\sin 54^\circ} = \cos 36^\circ = \tfrac{1}{4}(\sqrt{5}+1)$,

and $\mathbf{\cos 54^\circ} = \sin 36^\circ = \tfrac{1}{4}\sqrt{10-2\sqrt{5}}$.

(iv) $\because$ angle 72° is the complement of angle 18°,

$\therefore$ $\mathbf{\sin 72^\circ} = \cos 18^\circ = \tfrac{1}{4}\sqrt{10+2\sqrt{5}}$,

and $\mathbf{\cos 72^\circ} = \sin 18^\circ = \tfrac{1}{4}(\sqrt{5}-1)$.

Examples (5)

Ex. 1.✓ Find **sin 15°** and **cos 15°**.

$\cos 15^\circ + \sin 15^\circ = +\sqrt{1+\sin 30^\circ} = \sqrt{1+\tfrac{1}{2}} = \sqrt{\tfrac{3}{2}} \cdots \text{(i)}$

$\cos 15^\circ - \sin 15^\circ = +\sqrt{1-\sin 30^\circ} = \sqrt{1-\tfrac{1}{2}} = \sqrt{\tfrac{1}{2}} \cdots \text{(ii)}$

Adding (i) and (ii) we get $2\cos 15^\circ = \sqrt{\tfrac{3}{2}} + \sqrt{\tfrac{1}{2}} = \frac{\sqrt{3}+1}{\sqrt{2}}$.

$$\therefore \quad \mathbf{\cos 15^\circ} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Subtracting (ii) from (i) we get

$$2\sin 15^\circ = \sqrt{\tfrac{3}{2}} - \sqrt{\tfrac{1}{2}} = \frac{\sqrt{3}-1}{\sqrt{2}},$$

$$\therefore \quad \sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Ex. 2. Find $\sin\frac{\pi}{8}$ and $\cos\frac{\pi}{8}$.

$$\sin\frac{\pi}{8} = \sin 22\tfrac{1}{2}^\circ = +\sqrt{\tfrac{1}{2}(1-\cos 45^\circ)} = \sqrt{\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)}$$

$$= \sqrt{\frac{1}{2}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)} = \sqrt{\frac{1}{2}\times\frac{2-\sqrt{2}}{2}} = \tfrac{1}{2}\sqrt{2-\sqrt{2}}.$$

$$\cos\frac{\pi}{8} = \cos 22\tfrac{1}{2}^\circ = +\sqrt{\tfrac{1}{2}(1+\cos 45^\circ)} = \sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)}$$

$$= \tfrac{1}{2}\sqrt{2+\sqrt{2}}.$$

Ex. 3. Show that $\cos 7^\circ 30' = \frac{1}{4}(\sqrt{2}+\sqrt{3}-1)\sqrt{2+\sqrt{2}}$. [**Pat. '38**]

$$7^\circ 30' \times 2 = 15^\circ;$$

$$\because \quad 2\cos^2 7^\circ 30' = 1+\cos 15^\circ = 1+\frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{2\sqrt{2}+\sqrt{3}+1}{2\sqrt{2}},$$

$$\therefore \quad \cos^2 7^\circ 30' = \frac{2\sqrt{2}+\sqrt{3}+1}{4\sqrt{2}} = \tfrac{1}{8}(4+\sqrt{6}+\sqrt{2})$$

[multiplying the numerator and the denominator by $\sqrt{2}$.]

Again $\{\frac{1}{4}(\sqrt{2}+\sqrt{3}-1)\sqrt{2+\sqrt{2}}\}^2$

$$= \tfrac{1}{16}(6+2\sqrt{6}-2\sqrt{2}-2\sqrt{3})(2+\sqrt{2})$$

$$= \tfrac{1}{8}(3+\sqrt{6}-\sqrt{2}-\sqrt{3})(2+\sqrt{2}) = \tfrac{1}{8}(4+\sqrt{6}+\sqrt{2}).$$

$$\therefore \quad \cos^2 7^\circ 30' = \{\tfrac{1}{4}(\sqrt{2}+\sqrt{3}-1)\sqrt{2+\sqrt{2}}\}^2$$

$$\therefore \quad \cos 7^\circ 30' = \tfrac{1}{4}(\sqrt{2}+\sqrt{3}-1)\sqrt{2+\sqrt{2}}.$$

Ex. 4. Find the ratios of 3° and multiples of 3°.

$$\sin 3^\circ = \sin(18^\circ - 15^\circ) = \sin 18^\circ \cos 15^\circ - \cos 18^\circ \sin 15^\circ$$

$$= \tfrac{1}{4}(\sqrt{5}-1).\frac{\sqrt{3}+1}{2\sqrt{2}} - \tfrac{1}{4}\sqrt{10+2\sqrt{5}} \times \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$= \tfrac{1}{4}(\sqrt{5}-1).\frac{\sqrt{6}+\sqrt{2}}{4} - \tfrac{1}{4}.\sqrt{2}.\sqrt{5+\sqrt{5}} \times \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$= \tfrac{1}{16}(\sqrt{5}-1)(\sqrt{6}+\sqrt{2}) - \tfrac{1}{8}(\sqrt{3}-1)(\sqrt{5+\sqrt{5}}).$$

$$\cos 3^\circ = \cos(18^\circ - 15^\circ) = \cos 18^\circ \cos 15^\circ + \sin 18^\circ \sin 15^\circ$$

$$= \tfrac{1}{4}\sqrt{10+2\sqrt{5}} \times \frac{\sqrt{3}+1}{2\sqrt{2}} + \tfrac{1}{4}(\sqrt{5}-1) \times \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$= \tfrac{1}{8}(\sqrt{5+\sqrt{5}})(\sqrt{3}+1) + \tfrac{1}{16}(\sqrt{5}-1)(\sqrt{6}-\sqrt{2}).$$

N. B. With the help of the ratios of 3°, 15°, 18°, 30°, 36°, 45° obtained so far, the trigonometrical ratios of any multiple angle of 3° can be found, because $6^\circ = 36^\circ - 30^\circ$, $9^\circ = 45^\circ - 36^\circ$, $12^\circ = 30^\circ - 18^\circ$, $21^\circ = 36 - 15^\circ$; etc.

If the multiple of 3° be greater than 45°, its complement angle must be less than 45° ; so the ratios of the multiples of 3° greater than 45° can be obtained from this complement.

Ex. 5. Show that $\dfrac{1-\cos\theta}{\sin\theta} = \tan\dfrac{\theta}{2}$.

$$\text{L. H. S.} = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}.$$

Ex. 6. Prove that $(\cos^2 66^\circ - \sin^2 6^\circ)(\cos^2 48^\circ - \sin^2 12^\circ) = \frac{1}{16}$. [C. U. (B. Sc) '49]

$\because \cos 66^\circ = \cos(90^\circ - 24^\circ) = \sin 24^\circ$,

and $\cos 48^\circ = \cos(90^\circ - 42^\circ) = \sin 42^\circ$,

$\therefore$ the given $\text{L.H.S.} = (\sin^2 24^\circ - \sin^2 6^\circ)(\sin^2 42^\circ - \sin^2 12^\circ)$

$= \sin(24^\circ + 6^\circ)\sin(24^\circ - 6^\circ)\sin(42^\circ + 12^\circ)\sin(42^\circ - 12^\circ)$

[$\because \sin^2 A - \sin^2 B = \sin(A+B).\sin(A-B)$]

$= \sin 30^\circ \sin 18^\circ \sin 54^\circ \sin 30^\circ$

$= \frac{1}{2}.\frac{1}{4}(\sqrt{5}-1).\ \frac{1}{4}(\sqrt{5}+1).\ \frac{1}{2} = \frac{1}{64}(\sqrt{5}+1)(\sqrt{5}-1)$

$= \frac{1}{64} \times 4 = \frac{1}{16}.$

Ex. 7. Show that $4(\cos^3 10^\circ + \sin^3 20^\circ)$
$= 3(\cos 10^\circ + \sin 20^\circ)$.

$\because \quad \cos 3A = 4\cos^3 A - 3\cos A$,

$\therefore \quad \cos 30^\circ = 4\cos^3 10^\circ - 3\cos 10^\circ$ [Putting 10° for A]

Again, $\because \quad \sin 3A = 3\sin A - 4\sin^3 A$,

$\therefore \quad \sin 60^\circ = 3\sin 20^\circ - 4\sin^3 20^\circ$ [Putting 20° for A]

Now, $4(\cos^3 10^\circ + \sin^3 20^\circ) = 4\cos^3 10^\circ + 4\sin^3 20^\circ$

$= \cos 30^\circ + 3\cos 10^\circ + 3\sin 20^\circ - \sin 60^\circ$

$= \dfrac{\sqrt{3}}{2} + 3(\cos 10^\circ + \sin 20^\circ) - \dfrac{\sqrt{3}}{2}$

$= 3(\cos 10^\circ + \sin 20^\circ)$.

Ex. 8. If $\cos A = \frac{4}{5}$ and $\cos B = \frac{3}{5}$, find the value of $\cos \dfrac{A-B}{2}$, A and B being positive acute angles.

$\because \quad \cos \dfrac{\theta}{2} = \pm \sqrt{\tfrac{1}{2}(1 + \cos\theta)}$,

$\therefore \quad \cos \dfrac{A-B}{2} = \sqrt{\tfrac{1}{2}\{1 + \cos(A-B)\}}$ [Putting $A-B$ for θ]

$\because \quad \cos A = \frac{4}{5}, \quad \therefore \quad \sin A = \sqrt{1 - \cos^2 A}$

$= \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$ [$\because$ A is acute]

and $\because$ $\cos B = \frac{3}{5}$, $\therefore$ $\sin B = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$ [$\because$ B is acute]

Now, $\cos(A-B) = \cos A \cos B + \sin A \sin B$

$\therefore \quad \cos(A-B) = \frac{4}{5} \times \frac{3}{5} + \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$.

$\therefore \quad \cos \dfrac{A-B}{2} = \sqrt{\tfrac{1}{2}(1 + \tfrac{24}{25})} = \sqrt{\tfrac{1}{2} \times \tfrac{49}{25}} = \tfrac{7}{5}\sqrt{\tfrac{1}{2}} = \dfrac{7}{5\sqrt{2}}$.

✓**Ex. 9.** If $270^\circ < \theta < 360^\circ$ and $\cos\theta = \frac{119}{169}$, find $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$.

$\because$ θ is greater than 270° but less than 360°,

$\therefore$ $\frac{\theta}{2}$ must be greater than 135° but less than 180°.

$\therefore$ here $\sin\frac{\theta}{2}$ is positive and $\cos\frac{\theta}{2}$ is negative.

$$\text{Now, } \sin\frac{\theta}{2} = +\sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\frac{119}{169}}{2}} = \sqrt{\frac{25}{169}} = \frac{5}{13},$$

$$\text{and } \cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\frac{119}{169}}{2}} = -\sqrt{\frac{144}{169}} = -\frac{12}{13}.$$

Ex. 10. Show that $\dfrac{2\sin A - \sin 2A}{2\sin A + \sin 2A} = \tan^2\dfrac{A}{2}$.

$$\text{L. H. S.} = \frac{2\sin A - 2\sin A\cos A}{2\sin A + 2\sin A\cos A} = \frac{2\sin A(1-\cos A)}{2\sin A(1+\cos A)}$$

$$= \frac{1-\cos A}{1+\cos A} = \frac{2\sin^2\frac{A}{2}}{2\cos^2\frac{A}{2}} = \tan^2\frac{A}{2}.$$

Ex. 11. Prove that $(\cos A + \cos B)^2 + (\sin A - \sin B)^2$

$$= 4\cos^2\frac{A+B}{2}.$$

$$\text{L. H. S.} = \cos^2 A + \cos^2 B + 2\cos A\cos B + \sin^2 A + \sin^2 B - 2\sin A\sin B$$

$$= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) + 2(\cos A\cos B - \sin A\sin B)$$

$$= 1 + 1 + 2\cos(A+B) = 2 + 2\cos(A+B)$$

$$= 2\{1+\cos(A+B)\} = 2\times 2\cos^2\frac{A+B}{2} = 4\cos^2\frac{A+B}{2}.$$

Ex. 12. If $A=240^\circ$, is the statement

$2\sin\frac{A}{2}=\sqrt{1+\sin A}-\sqrt{1-\sin A}$ correct ? If not how must it be modified ?

Here $A=240^\circ$, $\therefore$ $\frac{A}{2}=120^\circ$, so that $\sin\frac{A}{2}$ is positive and greater than $\cos\frac{A}{2}$.

$$\therefore\ \sin\frac{A}{2}+\cos\frac{A}{2}=+\sqrt{1+\sin A}\cdots\cdots(1)$$

$$\text{and } \sin\frac{A}{2}-\cos\frac{A}{2}=+\sqrt{1-\sin A}\cdots\cdots(2)$$

$\therefore$ Adding (1) & (2) we get $2\sin\frac{A}{2}=\sqrt{1+\sin A}+\sqrt{1-\sin A}.$

So, the statement here is not correct. It will be correct, if it is modified as $2\sin\frac{A}{2}=\sqrt{1+\sin A}+\sqrt{1-\sin A}.$

Ex. 13. If $\sec(\phi+\alpha)+\sec(\phi-\alpha)=2\sec\phi$, prove that $\cos\phi=\sqrt{2}\cos\frac{\alpha}{2}$. [Pat. '44]

From the given condition we have

$$\frac{1}{\cos(\phi+\alpha)}+\frac{1}{\cos(\phi-\alpha)}=\frac{2}{\cos\phi},$$

$$\text{or, } \frac{\cos(\phi-\alpha)+\cos(\phi+\alpha)}{\cos(\phi+\alpha)\cos(\phi-\alpha)}=\frac{2}{\cos\phi},$$

$$\text{or, } \frac{2\cos\phi\cos\alpha}{\cos^2\phi-\sin^2\alpha}=\frac{2}{\cos\phi},$$

$$\text{or, } 2\cos^2\phi-2\sin^2\alpha=2\cos^2\phi\cos\alpha,$$

$$\text{or, } 2\cos^2\phi-2\cos^2\phi\cos\alpha=2\sin^2\alpha,$$

$$\text{or, } 2\cos^2\phi(1-\cos\alpha)=2\sin^2\alpha,$$

$$\text{or, } \cos^2\phi=\frac{\sin^2\alpha}{1-\cos\alpha}=\frac{1-\cos^2\alpha}{1-\cos\alpha}=1+\cos\alpha=2\cos^2\frac{\alpha}{2},$$

$$\therefore\ \cos\phi=\sqrt{2}\cos\frac{\alpha}{2}.$$

Ex. 14. If $\tan A=\dfrac{\sin B \sin C}{\cos B+\cos C}$, prove that one of the values of $\tan\dfrac{A}{2}$ is $\tan\dfrac{B}{2}\tan\dfrac{C}{2}$.

$\because\ \sec^2 A=1+\tan^2 A,$

$$\therefore\quad \text{here } \sec^2 A=1+\left(\frac{\sin B \sin C}{\cos B+\cos C}\right)^2$$

$$=1+\frac{\sin^2 B \sin^2 C}{(\cos B+\cos C)^2}=\frac{(\cos B+\cos C)^2+\sin^2 B \sin^2 C}{(\cos B+\cos C)^2}$$

$$=\frac{(\cos B+\cos C)^2+(1-\cos^2 B)(1-\cos^2 C)}{(\cos B+\cos C)^2}$$

$$=\frac{\cos^2 B \cos^2 C+2 \cos B \cos C+1}{(\cos B+\cos C)^2}=\frac{(1+\cos B \cos C)^2}{(\cos B+\cos C)^2}$$

$$\therefore\quad \sec A=\frac{1+\cos B \cos C}{\cos B+\cos C},\quad \therefore\quad \cos A=\frac{\cos B+\cos C}{1+\cos B \cos C}.$$

$\therefore$ By Comp. & Div. we have

$$\frac{1-\cos A}{1+\cos A}=\frac{1+\cos B \cos C-\cos B-\cos C}{1+\cos B \cos C+\cos B+\cos C}$$

$$=\frac{(1-\cos B)(1-\cos C)}{(1+\cos B)(1+\cos C)},$$

$$\therefore\quad \frac{2 \sin^2 \frac{A}{2}}{2 \cos^2 \frac{A}{2}}=\frac{2 \sin^2 \frac{B}{2}.\, 2 \sin^2 \frac{C}{2}}{2 \cos^2 \frac{B}{2}.\, 2 \cos^2 \frac{C}{2}}=\frac{\sin^2 \frac{B}{2} \sin^2 \frac{C}{2}}{\cos^2 \frac{B}{2} \cos^2 \frac{C}{2}}$$

$$\therefore\quad \tan^2 \frac{A}{2}=\tan^2 \frac{B}{2} \tan^2 \frac{C}{2},\quad \therefore\quad \tan\frac{A}{2}=\tan\frac{B}{2}\tan\frac{C}{2}.$$

Hence one value of $\tan\dfrac{A}{2}$ is $\tan\dfrac{B}{2}\tan\dfrac{C}{2}$.

Exercise 5

1. Find $\sin 9^\circ$ and $\cos 9^\circ$.

2. Prove that $\cos 15^\circ - \sin 15^\circ = \frac{1}{\surd 2}$. [B. H. U. '38]

3. Find the value of $\cos^2 48^\circ - \sin^2 12^\circ$.

4. Find the value of $\cos^2 36^\circ + \sin^2 18^\circ$.

5. Evaluate $2 \sin 75^\circ \sin 15^\circ$.

6. Show that $\frac{1+\cos\theta}{\sin\theta} = \cot\frac{\theta}{2}$.

7. Show that $\frac{\cos A}{1-\sin A} = \frac{\cot\frac{1}{2}A + 1}{\cot\frac{1}{2}A - 1}$.

8. If $\sin A = \frac{60}{61}$ and $\sin B = \frac{4}{5}$, find the value of $\sin^2\frac{A-B}{2}$ and $\cos^2\frac{A+B}{2}$, the angles A and B being positive acute angles.

9. Show that $\cos^2 18^\circ \sin^2 36^\circ + \cos 36^\circ \sin 18^\circ = \frac{9}{16}$.

10. Find the value of $\sin^2 72^\circ \cos^2 54^\circ - \sin 54^\circ \cos 72^\circ$. [C. U. (B. Sc.) '48]

11. If $\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$, find the value of $\cos(\alpha+\beta)$ and $\tan\frac{\alpha-\beta}{2}$.

12. If A lies between 450° and 630°, find $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ in terms of A.

Prove that :—

13. $\sec\theta + \tan\theta = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$. [C. U. '39]

14. $4(\cos^3 25^\circ + \cos^3 35^\circ) = 3(\cos 25^\circ + \cos 35^\circ)$.

15. $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ = 1.$

16. $\tan \dfrac{A+B}{2} + \tan \dfrac{A-B}{2} = \dfrac{2 \sin A}{\cos A + \cos B}.$ [B. H. U. '39]

17. $\cos \frac{1}{2}(\phi - \theta) - \sin \theta \sin \frac{1}{2}(\phi + \theta) = \cos \theta \cos \frac{1}{2}(\phi + \theta).$ [C. U. '50]

18. $16 \cos \dfrac{2\pi}{15} \cos \dfrac{4\pi}{15} \cos \dfrac{8\pi}{15} \cos \dfrac{14\pi}{15} = 1.$ [B. H. U. '47]

19. $\cos^4 \dfrac{\pi}{8} + \cos^4 \dfrac{3\pi}{8} + \cos^4 \dfrac{5\pi}{8} + \cos^4 \dfrac{7\pi}{8} = \dfrac{3}{2}.$ [Pat. '38, B. H. U. '46]

20. $(\cos x + \cos y)^2 + (\sin x + \sin y)^2 = 4 \cos^2 \frac{1}{2}(x - y).$

21. If $\theta = 340^\circ$, is the statement $2 \sin \dfrac{\theta}{2} = -\sqrt{1 + \sin \theta} - \sqrt{1 - \sin \theta}$ correct ? If not, how must it be modified ?

22. If $A = 320^\circ$, prove that $\tan \dfrac{A}{2} = \dfrac{-1 + \sqrt{1 + \tan^2 A}}{\tan A}.$

23. If $\cos \theta = \dfrac{\cos \alpha - \cos \beta}{1 - \cos \alpha \cos \beta}$, prove that one value of $\tan \dfrac{\theta}{2}$ is $\tan \dfrac{\alpha}{2} \cot \dfrac{\beta}{2}.$ [Pat. '42]

24. If $\sin \alpha = -\frac{4}{5}$, and α lies between 180° and 270°, find the values of $\sin \dfrac{\alpha}{2}$ and $\cos \dfrac{\alpha}{2}.$ [Pat. '42]

25. Prove that $2 \sin \dfrac{A}{2} = \pm \sqrt{1 + \sin A} \pm \sqrt{1 - \sin A}$ and determine which are the correct signs when $270^\circ > A > 180^\circ.$ [B. H. U. '31]

26. If $\tan\frac{\theta}{2}=\sqrt{\frac{1-e}{1+e}}.\tan\frac{\phi}{2}$, show that $\cos\phi=\frac{\cos\theta-e}{1-e\cos\theta}$.

[Pat. '40 ; A. U. '44, '46]

[Hints : $\tan\frac{\phi}{2}=\frac{\sqrt{1+e}\sin\frac{\theta}{2}}{\sqrt{1-e}\cos\frac{\theta}{2}}$, or, $\frac{1}{\tan^2\frac{\phi}{2}}=\frac{(1-e)\cos^2\frac{\theta}{2}}{(1+e)\sin^2\frac{\theta}{2}}$

$$\therefore \frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}=\frac{(1-e)\cos^2\frac{\theta}{2}-(1+e)\sin^2\frac{\theta}{2}}{(1-e)\cos^2\frac{\theta}{2}+(1+e)\sin^2\frac{\theta}{2}}$$

Again, $\cos\phi=\frac{2-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}$, $\therefore\ \cos\phi=\cdots$]

27. Show that

$$\sin x=2^n\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cdots\cdots\cos\frac{x}{2^n}.\sin\frac{x}{2^n}.$$

Trigonometrical Identities

29. Many interesting identities can be established connecting the functions of three or more angles when there is a relation among them, especially when the three angles A, B, C satisfy the relation $A+B+C=180^\circ\ (\pi)$. In proving these identities we shall have to use the properties of complementary and supplementary angles.

(1) If $A+B+C=\pi=180^\circ$, the sum of any two of them is the supplement of the third. Thus, when $A+B+C=180^\circ$, $A+B=180^\circ-C=\pi-C$, $B+C=180^\circ-A$, $A+C=180-B$.

Hence, (i) $\sin(A+B)=\sin(\pi-C)=\sin C$,

(ii) $\cos(A+B)=\cos(\pi-C)=-\cos C$,

(iii) $\sin C=\sin(A+B)$.

(iv) $\cos C = -\cos(A+B)$,

(v) $\tan(A+B) = \tan(\pi - C) = -\tan C$,

(vi) $\cot(A+B) = -\cot C$.

(2) If $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=90^\circ=\frac{\pi}{2}$, or, if $A+B+C=180^\circ$, then each of $\frac{A}{2}$, $\frac{B}{2}$, $\frac{C}{2}$ will be the complement of the other two,

i. e., $\frac{A}{2}+\frac{B}{2}=90^\circ-\frac{C}{2}=\frac{\pi}{2}-\frac{C}{2}$, $\frac{B}{2}+\frac{C}{2}=90^\circ-\frac{A}{2}$

and $\frac{C}{2}+\frac{A}{2}=90^\circ-\frac{B}{2}$.

Hence, (i) $\sin\left(\frac{A}{2}+\frac{B}{2}\right)=\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cos\frac{C}{2}$,

(ii) $\cos\left(\frac{A}{2}+\frac{B}{2}\right)=\sin\frac{C}{2}$,

(iii) $\tan\left(\frac{A}{2}+\frac{B}{2}\right)=\cot\frac{C}{2}$,

(iv) $\sin\frac{A}{2}=\cos\frac{B+C}{2}$, $\cos\frac{A}{2}=\sin\frac{B+C}{2}$,

$\tan\frac{A}{2}=\cot\frac{B+C}{2}$, etc.

Examples (6)

Ex. 1. If $A+B+C=\pi$, prove that

$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$.

[C. U. '33, '37, '38, '53, C. Pre-U. '64]

L. H. S. $=(\sin 2A+\sin 2B)+\sin 2C$

$=2\sin(A+B)\cos(A-B)+2\sin C\cos C$

$=2\sin C\cos(A-B)+2\sin C\cos C$

[$\because$ $A+B=180^\circ-C$, $\therefore$ $\sin(A+B)=\sin C$]

$$=2\sin C\{\cos(A-B)+\cos C\}$$

$$=2\sin C\{\cos(A-B)-\cos(A+B)\}$$

$$[\because \cos C=-\cos(A+B)]$$

$$=2\sin C\times 2\sin A\sin B=4\sin A\sin B\sin C.$$

Ex. 2. **If $A+B+C=\pi$, prove that**

$\cos 2A+\cos 2B+\cos 2C=-4\cos A\cos B\cos C-1.$

L. H. S. $=(\cos 2A+\cos 2B)+\cos 2C$

$$=2\cos(A+B)\cos(A-B)+2\cos^2 C-1$$

$$=-2\cos C\cos(A-B)+2\cos^2 C-1$$

$$[\because \cos(A+B)=\cos(\pi-C)=-\cos C]$$

$$=-2\cos C\{\cos(A-B)-\cos C\}-1$$

$$=-2\cos C\{\cos(A-B)+\cos(A+B)\}-1$$

$$[\because -\cos C=\cos(A+B)]$$

$$=-2\cos C\times 2\cos A\cos B-1$$

$$=-4\cos A\cos B\cos C-1.$$

Ex. 3. If $A+B+C=180^\circ$, show that $\tan 2A+\tan 2B+\tan 2C=\tan 2A\tan 2B\tan 2C$.

$$\tan(2A+2B)=\frac{\tan 2A+\tan 2B}{1-\tan 2A\tan 2B}.$$

Now, $\because 2A+2B+2C=360^\circ$,

$\therefore\ 2A+2B=360^\circ-2C,\ \therefore\ \tan(2A+2B)=\tan(360^\circ-2C)$

$$=-\tan 2C.$$

Hence, $-\tan 2C=\dfrac{\tan 2A+\tan 2B}{1-\tan 2A\tan 2B}$

$\therefore\ \tan 2A+\tan 2B=-\tan 2C+\tan 2A\tan 2B\tan 2C$

[By cross-multiplication]

$\therefore\ \tan 2A+\tan 2B+\tan 2C=\tan 2A\tan 2B\tan 2C.$

Ex. 4. **If $A+B+C=\pi$, prove that**

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$ [C. U. '29, 50]

L. H. S. $=(\sin A+\sin B)+\sin C$

$$=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$=2\cos\frac{C}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}\left[\because\ \frac{A+B}{2}=90^\circ-\frac{C}{2}\right]$$

$$=2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}+\sin\frac{C}{2}\right)$$

$$=2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right)$$

$$\left[\because\ \frac{A}{2}+\frac{B}{2}=90^\circ-\frac{C}{2}\right]$$

$$=2\cos\frac{C}{2}\times 2\cos\frac{A}{2}\cos\frac{B}{2}=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Ex. 5. **If $A+B+C=\pi$, prove that**

$$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

L. H. S. $=(\cos A+\cos B)+\cos C$

$$=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2}$$

$$\left[\because\ \cos C=1-2\sin^2\frac{C}{2}\right]$$

$$=2\sin\frac{C}{2}\cos\frac{A-B}{2}-2\sin^2\frac{C}{2}+1$$

$$\left[\because\ \frac{A+B}{2}=90^\circ-\frac{C}{2},\ \therefore\ \cos\frac{A+B}{2}=\sin\frac{C}{2}\right]$$

$$=2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\sin\frac{C}{2}\right)+1$$

$$=2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)+1$$

$$=2\sin\frac{C}{2}\times 2\sin\frac{A}{2}\sin\frac{B}{2}+1$$

$$=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+1.$$

Ex. 6. If $A+B+C=\pi$, prove that

$\tan A+\tan B+\tan C=\tan A\tan B\tan C$. [C. U.]

$\because\ A+B+C=\pi,\ \therefore\ A+B=\pi-C,$

$\therefore\ \tan(A+B)=\tan(\pi-C)=-\tan C$

$$\therefore\ \frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C,$$

or, $\tan A+\tan B=-\tan C+\tan A\tan B\tan C$

[by cross-multiplication]

$\therefore\ \tan A+\tan B+\tan C=\tan A\tan B\tan C.$

Ex. 7. If $A+B+C=\pi$, show that

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=1+4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}.$$

Now, $4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}$

$$=2\sin\frac{\pi-A}{4}\left(2\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}\right)$$

$$=2\sin\frac{\pi-A}{4}\left\{\cos\frac{B-C}{4}-\cos\frac{2\pi-(B+C)}{4}\right\}$$

$$=2\sin\frac{\pi-A}{4}\left\{\cos\frac{B-C}{4}-\cos\frac{2\pi-(\pi-A)}{4}\right\}$$

= ?

$$=2\sin\frac{\pi-A}{4}\left\{\cos\frac{B-C}{4}-\cos\frac{\pi+A}{4}\right\}$$

$$=2\sin\frac{\pi-A}{4}\cos\frac{B-C}{4}-2\sin\frac{\pi-A}{4}\cos\frac{\pi+A}{4}$$

$$=\left(\sin\frac{\pi-A+B-C}{4}+\sin\frac{\pi-A-B+C}{4}\right)-\left(\sin\frac{\pi}{2}+\sin\frac{-2A}{4}\right)$$

$$=\sin\frac{2B}{4}+\sin\frac{2C}{4}-\sin\frac{\pi}{2}-\left(-\sin\frac{A}{2}\right)$$

$$=\sin\frac{B}{2}+\sin\frac{C}{2}-1+\sin\frac{A}{2}\left[\because\ \sin\frac{\pi}{2}=1\right]$$

$$\therefore\ \sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=1+4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}.$$

Ex. 8. If $A+B+C=\pi$, prove that

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}.$$

Here the R. H. S. $=2\cos\frac{\pi-A}{4}\left(2\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}\right)$

$$=2\cos\frac{\pi-A}{4}\left\{\cos\frac{2\pi-(B+C)}{4}+\cos\frac{B-C}{4}\right\}$$

$$=2\cos\frac{\pi-A}{4}\left\{\cos\frac{2\pi-(\pi-A)}{4}+\cos\frac{B-C}{4}\right\}$$

$$=2\cos\frac{\pi-A}{4}\left(\cos\frac{\pi+A}{4}+\cos\frac{B-C}{4}\right)$$

$$=2\cos\frac{\pi-A}{4}\cos\frac{\pi+A}{4}+2\cos\frac{\pi-A}{4}\cos\frac{B-C}{4}$$

$$=\left(\cos\frac{\pi}{2}+\cos\frac{A}{2}\right)+2\cos\frac{B+C}{4}\cos\frac{B-C}{4}$$

$$=\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\left[\because\ \cos\frac{\pi}{2}=0\right]$$

Ex. 9. If $A+B+C=\pi$, prove that

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1.$$

[C. U. '36, '39]

$\because\ A+B+C=\pi=180^\circ,\ \therefore\ \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=90^\circ,$

or, $\frac{A}{2}+\frac{B}{2}=90^\circ-\frac{C}{2}$

$\therefore\ \tan\left(\frac{A}{2}+\frac{B}{2}\right)=\tan\left(90^\circ-\frac{C}{2}\right)=\cot\frac{C}{2},$

or, $\dfrac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\cot\frac{C}{2}=\dfrac{1}{\tan\frac{C}{2}},$

or, $\tan\frac{A}{2}\tan\frac{C}{2}+\tan\frac{B}{2}\tan\frac{C}{2}=1-\tan\frac{A}{2}\tan\frac{B}{2}$

[by cross-multiplication]

$\therefore\ \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1.$

Ex. 10. If $A+B+C=\pi$, prove that
$\sin^2A+\sin^2B+\sin^2C=2+2\cos A\cos B\cos C.$

[C. U. ; Pat. U.]

$\because\ \cos 2A=1-2\sin^2A,\quad \therefore\ 2\sin^2A=1-\cos 2A$

$\therefore\ \sin^2A=\frac{1}{2}(1-\cos 2A).$ Similarly, $\sin^2B=\frac{1}{2}(1-\cos 2B).$

Now, $\sin^2A+\sin^2B+\sin^2C$

$=\frac{1}{2}(1-\cos 2A)+\frac{1}{2}(1-\cos 2B)+\sin^2C$

$=\frac{1}{2}+\frac{1}{2}-\frac{1}{2}(\cos 2A+\cos 2B)+\sin^2C$

$=1-\frac{1}{2}\times 2\cos(A+B)\cos(A-B)+1-\cos^2C$

$=2-\cos(A+B)\cos(A-B)-\cos^2C$

$$=2+\cos C\cos(A-B)-\cos^2 C$$

$$[\because \cos(A+B)=\cos(\pi-C)=-\cos C]$$

$$=2+\cos C\{\cos(A-B)-\cos C\}$$

$$=2+\cos C\{\cos(A-B)+\cos(A+B)\}$$

$$=2+\cos C\times 2\cos A\cos B=2+2\cos A\cos B\cos C.$$

Ex. 11. If $A+B+C=\pi$, prove that

$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1.$

[C. U. '37, '47]

$$\because \cos 2A=2\cos^2 A-1,$$

$$\therefore 2\cos^2 A=1+\cos 2A,$$

$$\therefore \cos^2 A=\tfrac{1}{2}(1+\cos 2A).$$

Similarly, $\cos^2 B=\frac{1}{2}(1+\cos 2B)$.

Now, $\cos^2 A+\cos^2 B+\cos^2 C$

$$=\tfrac{1}{2}(1+\cos 2A+1+\cos 2B)+\cos^2 C$$

$$=1+\tfrac{1}{2}(\cos 2A+\cos 2B)+\cos^2 C$$

$$=1+\tfrac{1}{2}\times 2\cos(A+B)\cos(A-B)+\cos^2 C$$

$$=1+\cos(A+B)\cos(A-B)+\cos^2 C$$

$$=1-\cos C\cos(A-B)+\cos^2 C$$

$$[\because A+B=\pi-C, \quad \therefore \cos(A+B)=-\cos C]$$

$$=1-\cos C\{\cos(A-B)-\cos C\}$$

$$=1-\cos C\{\cos(A-B)+\cos(A+B)\}$$

$$=1-\cos C\times 2\cos A\cos B=1-2\cos A\cos B\cos C.$$

$$\therefore \cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1.$$

Ex. 12. If $A+B+C=180^\circ$, show that

$\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$

L. H. S. $=(\sin A+\sin B)-\sin C$

$$=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}-2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$=2\cos\frac{C}{2}\cos\frac{A-B}{2}-2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$\left[\because\ \frac{A+B}{2}=\frac{\pi}{2}-\frac{C}{2}\right]$$

$$=2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}-\sin\frac{C}{2}\right)$$

$$=2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)$$

$$=2\cos\frac{C}{2}\times 2\sin\frac{A}{2}\sin\frac{B}{2}=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$$

Ex. 13. If $A+B+C=\frac{\pi}{2}$, prove that

$\sin^2 A+\sin^2 B+\sin^2 C+2\sin A\sin B\sin C=1$. [C. U. '38]

$\because\ \cos 2A=1-2\sin^2 A,\quad \therefore\ 2\sin^2 A=1-\cos 2A.$

$\therefore\ \sin^2 A=\frac{1}{2}(1-\cos 2A)$. Similarly $\sin^2 B=\frac{1}{2}(1-\cos 2B)$.

Now, $\sin^2 A+\sin^2 B+\sin^2 C$

$=\frac{1}{2}(1-\cos 2A+1-\cos 2B)+\sin^2 C$

$=1-\frac{1}{2}(\cos 2A+\cos 2B)+\sin^2 C$

$=1-\frac{1}{2}\times 2\cos(A+B)\cos(A-B)+\sin C.\sin C$

$=1-\sin C\cos(A-B)+\sin C.\cos(A+B)$

$\left[\because\ A+B=\frac{\pi}{2}-C,\quad \therefore\ \cos(A+B)=\cos\left(\frac{\pi}{2}-C\right)=\sin C.\right]$

$=1-\sin C\{\cos(A-B)-\cos(A+B)\}$

$=1-\sin C\times 2\sin A\sin B=1-2\sin A\sin B\sin C.$

$\therefore\ \sin^2 A+\sin^2 B+\sin^2 C+2\sin A\sin B\sin C=1.$

$$=2+\cos C\cos(A-B)-\cos^2 C$$

$$[\because \cos(A+B)=\cos(\pi-C)=-\cos C]$$

$$=2+\cos C\{\cos(A-B)-\cos C\}$$

$$=2+\cos C\{\cos(A-B)+\cos(A+B)\}$$

$$=2+\cos C\times 2\cos A\cos B=2+2\cos A\cos B\cos C.$$

Ex. 11. If $A+B+C=\pi$, prove that

$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1.$

[C. U. '37, '47]

$\because \cos 2A=2\cos^2 A-1,$

$\therefore 2\cos^2 A=1+\cos 2A,$

$\therefore \cos^2 A=\frac{1}{2}(1+\cos 2A).$

Similarly, $\cos^2 B=\frac{1}{2}(1+\cos 2B)$.

Now, $\cos^2 A+\cos^2 B+\cos^2 C$

$$=\tfrac{1}{2}(1+\cos 2A+1+\cos 2B)+\cos^2 C$$

$$=1+\tfrac{1}{2}(\cos 2A+\cos 2B)+\cos^2 C$$

$$=1+\tfrac{1}{2}\times 2\cos(A+B)\cos(A-B)+\cos^2 C$$

$$=1+\cos(A+B)\cos(A-B)+\cos^2 C$$

$$=1-\cos C\cos(A-B)+\cos^2 C$$

$$[\because A+B=\pi-C,\quad \therefore \cos(A+B)=-\cos C]$$

$$=1-\cos C\{\cos(A-B)-\cos C\}$$

$$=1-\cos C\{\cos(A-B)+\cos(A+B)\}$$

$$=1-\cos C\times 2\cos A\cos B=1-2\cos A\cos B\cos C.$$

$\therefore \cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1.$

Ex. 12. If $A+B+C=180^\circ$, show that

$\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$

L. H. S. $=(\sin A+\sin B)-\sin C$

$$=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}-2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$=2\cos\frac{C}{2}\cos\frac{A-B}{2}-2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$\left[\because \frac{A+B}{2}=\frac{\pi}{2}-\frac{C}{2}\right]$$

$$=2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}-\sin\frac{C}{2}\right)$$

$$=2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)$$

$$=2\cos\frac{C}{2}\times 2\sin\frac{A}{2}\sin\frac{B}{2}=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$$

Ex. 13. If $A+B+C=\frac{\pi}{2}$, prove that

$\sin^2A+\sin^2B+\sin^2C+2\sin A\sin B\sin C=1.$ [C. U. '38]

$\because \cos 2A=1-2\sin^2A, \quad \therefore \quad 2\sin^2A=1-\cos 2A.$

$\therefore \quad \sin^2A=\frac{1}{2}(1-\cos 2A).$ Similarly $\sin^2B=\frac{1}{2}(1-\cos 2B).$

Now, $\sin^2A+\sin^2B+\sin^2C$

$=\frac{1}{2}(1-\cos 2A+1-\cos 2B)+\sin^2C$

$=1-\frac{1}{2}(\cos 2A+\cos 2B)+\sin^2C$

$=1-\frac{1}{2}\times 2\cos(A+B)\cos(A-B)+\sin C.\sin C$

$=1-\sin C\cos(A-B)+\sin C.\cos(A+B)$

$\left[\because A+B=\frac{\pi}{2}-C, \quad \therefore \quad \cos(A+B)=\cos\left(\frac{\pi}{2}-C\right)=\sin C.\right]$

$=1-\sin C\{\cos(A-B)-\cos(A+B)\}$

$=1-\sin C\times 2\sin A\sin B=1-2\sin A\sin B\sin C.$

$\therefore \quad \sin^2A+\sin^2B+\sin^2C+2\sin A\sin B\sin C=1.$

Ex. 14. If $A+B+C=180^\circ$, shew that

$$\cos\frac{A}{2}\cos\frac{B-C}{2}+\cos\frac{B}{2}\cos\frac{C-A}{2}+\cos\frac{C}{2}\cos\frac{A-B}{2}$$
$$=\sin A+\sin B+\sin C.$$

$\because\ A+B+C=180^\circ,\ \therefore\ \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=90^\circ,$

$\therefore\ \cos\frac{A}{2}=\cos\left\{90^\circ-\left(\frac{B+C}{2}\right)\right\}=\sin\frac{B+C}{2}.$

Similarly, $\cos\frac{B}{2}=\sin\frac{C+A}{2}$, and $\cos\frac{C}{2}=\sin\frac{A+B}{2}$.

Now, the given L. H. S.

$=\sin\frac{B+C}{2}\cos\frac{B-C}{2}+\sin\frac{C+A}{2}\cos\frac{C-A}{2}+\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$=\frac{1}{2}(\sin B+\sin C)+\frac{1}{2}(\sin C+\sin A)+\frac{1}{2}(\sin A+\sin B)$

$=\sin A+\sin B+\sin C.$

Ex. 15. If $A+B+C=2\pi$, show that

$$\cos^2A+\cos^2B+\cos^2C-2\cos A\cos B\cos C=1.$$

$\because\ A+B+C=2\pi=360^\circ,\ \therefore\ A+B=360^\circ-C$

$\therefore\ \cos(A+B)=\cos(360^\circ-C)=\cos C.$

Now, $\cos^2A+\cos^2B+\cos^2C$

$=\frac{1}{2}(1+\cos 2A+1+\cos 2B)+\cos^2C$

$=1+\frac{1}{2}(\cos 2A+\cos 2B)+\cos^2C$

$=1+\frac{1}{2}\times 2\cos(A+B)\cos(A-B)+\cos^2C$

$=1+\cos(A+B)\cos(A-B)+\cos C.\cos C$

$=1+\cos C\cos(A-B)+\cos C\cos(A+B)$

$=1+\cos C\{\cos(A-B)+\cos(A+B)\}$

$=1+\cos C\times 2\cos A\cos B=1+2\cos A\cos B\cos C,$

$\therefore\ \cos^2A+\cos^2B+\cos^2C-2\cos A\cos B\cos C=1.$

Ex. 16. If $A+B+C=\pi$, show that

$$\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$ [See Ex. 1]

$$=4\times 2\sin\frac{A}{2}\cos\frac{A}{2}\times 2\sin\frac{B}{2}\cos\frac{B}{2}\times 2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$=32\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Again, $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ [See. Ex. 4]

$\therefore$ The given L. H. S.

$$=\frac{32\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Ex. 17. If $A+B+C=2\theta$, prove that

$$\cos^2\theta+\cos^2(\theta-A)+\cos^2(\theta-B)+\cos^2(\theta-C)$$
$$=2+2\cos A\cos B\cos C.$$

$\because$ $\cos 2\theta=2\cos^2\theta-1$, $\therefore$ $\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$.

Now, the L. H. S. $=\frac{1}{2}\{1+\cos 2\theta+1+\cos(2\theta-2A)$

$$+1+\cos(2\theta-2B)+1+\cos(2\theta-2C)\}$$

$$=\frac{1}{2}\left\{4+2\cos\frac{2\theta+2\theta-2A}{2}\cos\frac{2\theta-2\theta+2A}{2}+\right.$$

$$\left.2\cos\frac{4\theta-2B-2C}{2}\cos(B-C)\right\}$$

$$=2+\cos(2\theta-A)\cos A+\cos(2\theta-B-C)\cos(B-C)$$

$$=2+\cos(B+C)\cos A+\cos A\cos(B-C)$$

[$\because$ $2\theta-A=B+C$ and $2\theta-B-C=A$]

$$=2+\cos A\{\cos(B+C)+\cos(B-C)\}$$

$$=2+\cos A\times 2\cos B\cos C=2+2\cos A\cos B\cos C.$$

Ex. 18. If $\cos A+\cos B+\cos C=0$, show that

$\cos 3A+\cos 3B+\cos 3C=12\cos A\cos B\cos C$.

$$\text{L. H. S.}=4\cos^3 A-3\cos A+4\cos^3 B-3\cos B$$
$$+4\cos^3 C-3\cos C$$

$$=4(\cos^3 A+\cos^3 B+\cos^3 C)-3(\cos A+\cos B+\cos C)$$

$$=4\times 3\cos A\cos B\cos C-3\times 0$$

$$[\because \quad \cos A+\cos B+\cos C=0,$$
$$\therefore \quad \cos^3 A+\cos^3 B+\cos^3 C=3\cos A\cos B\cos C\ ]$$

$$=12\cos A\cos B\cos C.$$

Ex. 19. If $x+y+z=xyz$, prove that

$x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)=4xyz.$

Let $x=\tan A$, $y=\tan B$, and $z=\tan C$.

$\because \quad x+y+z=xyz,$

$\therefore \quad \tan A+\tan B+\tan C=\tan A\tan B\tan C,$

or, $\tan A-\tan A\tan B\tan C=-(\tan B+\tan C),$

or, $\tan A(1-\tan B\tan C)=-(\tan B+\tan C)$

or, $$\tan A=-\frac{\tan B+\tan C}{1-\tan B\tan C}=-\tan(B+C)$$
$$=\tan\{\pi-(B+C)\}$$

$\therefore \quad A=\pi-(B+C), \quad \therefore \quad A+B+C=\pi.$

$\therefore \quad \tan 2A+\tan 2B+\tan 2C=\tan 2A\tan 2B\tan 2C$

[See Ex. 3]

or, $$\frac{2\tan A}{1-\tan^2 A}+\frac{2\tan B}{1-\tan^2 B}+\frac{2\tan C}{1-\tan^2 C}$$

$$=\frac{8\tan A\tan B\tan C}{(1-\tan^2 A)(1-\tan^2 B)(1-\tan^2 C)},$$

or, $\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}$,

or, $\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$,

$\therefore\quad x(1-y^2)(1-z^2)+y(1-x^2)(1-z^2)+z(1-x^2)(1-y^2)$

$=4xyz$ [multiplying both sides by $(1-x^2)(1-y^2)(1-z^2)$]

Ex. 20. If $x+y+z=xyz$, prove that

$$\frac{3x-x^3}{1-3x^2}+\frac{3y-y^3}{1-3y^2}+\frac{3z-z^3}{1-3z^2}=\frac{3x-x^3}{1-3x^2}\cdot\frac{3y-y^3}{1-3y^2}\cdot\frac{3z-z^3}{1-3z^2}.$$

Let $x=\tan A$, $y=\tan B$, and $z=\tan C$.

$\because\quad x+y+z=xyz$,

$\therefore\quad \tan A+\tan B+\tan C=\tan A\tan B\tan C$,

$\therefore\quad \tan A+\tan B+\tan C-\tan A\tan B\tan C=0\cdots\cdots(1)$

Now, $\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$

$=0$ [$\because$ from (1) we get the numerator $=0$]

$=\tan 180°$.

$\therefore\quad A+B+C=n\pi$ (where n is any integer)

$\therefore\quad 3A+3B+3C=3n\pi$, or, $3A+3B=3n\pi-3C$,

$\therefore\quad \tan(3A+3B)=\tan(3n\pi-3C)=-\tan 3C$.

Again, $\tan(3A+3B)=\frac{\tan 3A+\tan 3B}{1-\tan 3A\tan 3B}$

$\therefore\quad \frac{\tan 3A+\tan 3B}{1-\tan 3A\tan 3B}=-\tan 3C$,

$\therefore\quad \tan 3A+\tan 3B=-\tan 3C+\tan 3A\tan 3B\tan 3C$,

$\therefore\quad \tan 3A+\tan 3B+\tan 3C=\tan 3A\tan 3B\tan 3C\cdots\cdots(2)$

Now, $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A} = \dfrac{3x - x^3}{1 - 3x^2}$ [$\because$ $x = \tan A$],

$\tan 3B = \dfrac{3 \tan B - \tan^3 B}{1 - 3 \tan^2 B} = \dfrac{3y - y^3}{1 - 3y^2}$ [$\therefore$ $y = \tan B$],

Similarly, $\tan 3C = \dfrac{3z - z^3}{1 - 3z^2}$. $\therefore$ from (2) we have

$$\frac{3x - x^2}{1 - 3x^2} + \frac{3y - y^3}{1 - 3y^2} + \frac{3z - z^3}{1 - 3z^2} = \frac{3x - x^3}{1 - 3x^2} \cdot \frac{3y - y^3}{1 - 3y^2} \cdot \frac{3z - z^3}{1 - 3z^2}.$$

Exercise 6

If $A + B + C = \pi$, prove that :

1. $\sin 2A - \sin 2B + \sin 2C = 4 \cos A \sin B \cos C$.

2. $\cos 2A + \cos 2B - \cos 2C + 4 \sin A \sin B \cos C = 1$.

3. $\cos A + \cos B - \cos C = 4 \cos \dfrac{A}{2} \cos \dfrac{B}{2} \sin \dfrac{C}{2} - 1$.

4. $\sin^2 A - \sin^2 B + \sin^2 C = 2 \sin A \sin C \cos B$.
[Pat. U. '40]

5. $\cot \dfrac{A}{2} + \cot \dfrac{B}{2} + \cot \dfrac{C}{2} = \cot \dfrac{A}{2} \cot \dfrac{B}{2} \cot \dfrac{C}{2}$.

6. $\dfrac{\sin B + \sin C - \sin A}{\sin A + \sin B + \sin C} = \tan \dfrac{B}{2} \tan \dfrac{C}{2}$.

7. $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$. [C. U. '55]

8. $\cos^2 A + \cos^2 B + 2 \cos A \cos B \cos C = \sin^2 C$. [C. U. '30]

9. $\dfrac{1 + \cos A - \cos B + \cos C}{1 + \cos A + \cos B - \cos C} = \tan \dfrac{B}{2} \cot \dfrac{C}{2}$.

10. $1 - 2 \sin B \sin A \cos C + \cos^2 C = \cos^2 A + \cos^2 B$.

11. $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$. [C. U. '48]

12. $\frac{\cos A}{\sin B\sin C}+\frac{\cos B}{\sin C\sin A}+\frac{\cos C}{\sin A\sin B}=2$. [C. U. '49]

13. $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=1+4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}$. [Pat. '39]

14. $\frac{\cot A+\cot B}{\tan A+\tan B}+\frac{\cot B+\cot C}{\tan B+\tan C}+\frac{\cot C+\cot A}{\tan C+\tan A}=1$.

15. $\sin(B+2C)+\sin(C+2A)+\sin(A+2B)$

$=4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}$. [H. S. '64]

16. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$. [Pat. '42]

17. If $A+B+C=\pi$, and $\cos A=\cos B\cos C$, show that $\tan A=\tan B+\tan C$. [C. U. '33]

If $A+B+C=\frac{\pi}{2}$, prove that :

18. $\cot A+\cot B+\cot C=\cot A\cot B\cot C$.

19. $\tan A\tan B+\tan B\tan C+\tan C\tan A=1$. [Pat '39]

20. $\frac{\cos A+\sin B+\sin C}{\sin A+\cos B+\sin C}=\frac{1-\tan\frac{1}{2}A}{1-\tan\frac{1}{2}B}$.

21. If $A+B=C$, prove that $\cos^2A+\cos^2B+\cos^2C$

$-2\cos A\cos B\cos C=1$. [Pat. '43]

22. If $A=B+C$, shew that $\sin(A+B+C)+\sin(A+B-C)$

$+\sin(A-B+C)=4\sin A\cos B\cos C$.

23. If $\alpha+\beta+\gamma=0$, prove that

$$1+2\sin\beta\sin\gamma\cos\alpha+\cos^2\alpha=\cos^2\beta+\cos^2\gamma.$$

24. If $A+B+C=2S$, shew that

$$\sin(S-A)\sin(S-B)+\sin S\sin(S-C)=\sin A\sin B.$$

25. If $A+B+C=180^\circ$; and $\sin\left(A+\frac{C}{2}\right)=n\sin\frac{C}{2}$, show that $\tan\frac{A}{2}\tan\frac{B}{2}=\frac{n-1}{n+1}$. [P. U. '45]

26. If $A+B+C+D=2\pi$, prove that

$$\frac{\tan A+\tan B+\tan C+\tan D}{\cot A+\cot B+\cot C+\cot D}=\tan A\tan B\tan C\tan D.$$

27. If $\cos(A+B)\sin(C+D)=\cos(A-B)\sin(C-D)$, show that $\cot A\cot B\cot C=\cot D$. [C. U. '30]

28. If α, β and θ be the angles of a triangle, show that $\cos^2 2\alpha+\cos^2 2\beta+\cos^2 2\theta=1+2\cos 2\alpha\cos 2\beta\cos 2\theta$.

29. If A, B, C and D be the angles of a quadrilateral, prove that

$$\cos A+\cos B+\cos C+\cos D+4\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A+C)\cos\tfrac{1}{2}(A+D)=0.$$

30. If $x+y+z=xyz$, prove that

$$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\cdot\frac{2y}{1-y^2}\cdot\frac{2z}{1-z^2}.$$

[U. P. B. '52]

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CHAPTER IV

GEOMETRY

Theorem 1

The angles made by a tangent to a circle with a chord drawn through the point of contact are respectively equal to the angles in the alternate segments of the circle.

Let PQ be a tangent to the circle BDR at the point B and let BD be a chord drawn through B, the point of contact.

To prove that

(i) $\angle PBD$ = the angle in the alternate segment,

and (ii) $\angle QBD$ = the angle in the alternate segment.

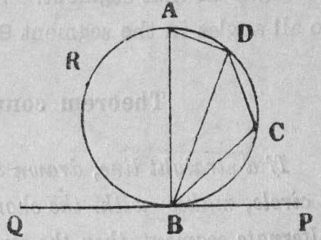


Fig. 1

Construction. From B draw the diameter BA meeting the circle at A.

Take any point C in the arc conjugate to the arc BRD. Join AD, DC, BC.

Proof. (i) $\because$ ADB is an angle in a semi-circle,

$\therefore$ $\angle ADB$ is a right angle.

$\therefore$ $\angle ABD + \angle BAD$ = one right angle.

Again, $\because$ PQ is a tangent at B and BA is a diameter of the circle through the point of contact, $\therefore$ $AB \perp PQ$.

$\therefore$ $\angle PBA$ = 1 right angle, *i.e.*, $\angle ABD + \angle PBD$ = 1 rt angle.

$\therefore$ $\angle ABD + \angle PBD = \angle ABD + \angle BAD$

$\therefore$ $\angle PBD = \angle BAD$, which is in the alternate segment.

(ii) ∵ ABCD is a cyclic quadrilateral,

∴ $\angle BCD + \angle BAD = 2$ right angles.

Also $\angle PBD + \angle QBD = 2$ right angles.

∴ $\angle PBD + \angle QBD = \angle BCD + \angle BAD$,

But $\angle PBD = \angle BAD$ [Proved]

∴ $\angle QBD = \angle BCD$, which is in the alternate segment.

Since angles in the same segment are equal, therefore $\angle PBD$ being equal to $\angle BAD$ in the segment BRD is equal to all angles in that segment. Similarly the $\angle QBD$ also is equal to all angles in the segment BCD.

Theorem converse to Theorem 1

If a straight line, drawn through an extremity of a chord of a circle, makes with the chord an angle equal to the angle in the alternate segment, then the straight line is a tangent to the circle.

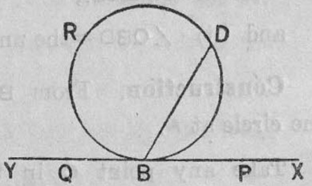


Fig. 2

Let BD be a chord of a circle BRD and let PQ, drawn through B, make $\angle PBD$ equal to the angle in the alternate segment BRD.

To prove that PQ is a tangent to the circle at B.

Construction. At B draw the tangent XY to the circle.

Proof. ∵ XY is a tangent to the circle at B and BD is a chord through the point of contact,

∴ $\angle XBD$ = the angle in the alternate segment BRD.

But $\angle PBD$ = the angle in the alternate segment BRD.

∴ $\angle PBD = \angle XBD$.

∴ PQ coincides with XY, which is the tangent at B.

∴ PQ is a tangent to the circle at B.

Miscellaneous Examples (1)

Ex. 1. Show that the perpendiculars dropped on the tangent and the chord through the point of contact, from the middle point of either arc cut off by the chord, are equal.

[C. U. 1915]

Let PD be a tangent and PQ a chord through the point of contact. C is the mid-point of arc PCQ. CD and CE are perpendiculars to PD and PQ respectively from C.

To prove that CD = CE.

Join CP and CQ.

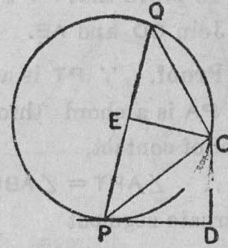


Fig. 3

Proof. $\because$ arc PC = arc QC,

$\therefore$ $\angle PQC = \angle QPC$.

Again, $\because$ PD is a tangent and PC a chord through the point of contact, $\therefore$ $\angle CPD = \angle PQC$ in the alt. segment.

$\therefore$ $\angle CPD = \angle QPC$. Now in $\triangle^s$ EPC and PCD, $\angle E = \angle D$ (right angles), $\angle EPC = \angle CPD$ and the side PC is common.

$\therefore$ the triangles are congruent. $\therefore$ CD = CE.

Ex. 2. AB, a diameter of a aircle, is produced to meet the tangent at C in D. Show that $\angle BDC + 2\angle BCD$ is a right angle.

Let O be the mid-point of AB, then O is the centre of the circle. Join OC and BC.

Proof. $\because$ OB = OC, $\therefore$ $\angle OCB = \angle OBC$.

The exterior $\angle OBC$ of $\triangle BCD = \angle BCD + \angle BDC$.

$\therefore$ $\angle OCB = \angle BCD + \angle BDC$. Adding $\angle BCD$ to both sides we have $\angle OCB + \angle BCD = 2\angle BCD + \angle BDC$. But CD is a tangent and CO a radius through the pt. of contact,

$\therefore$ the $\angle OCD$, i. e., $\angle OCB + \angle BCD =$ one right angle,

$\therefore$ $\angle BDC + 2\angle BCD =$ one right angle.

Ex. 3. Two circles intersect at A and B, and through P, any point on the circumference of one of them, straight lines PAC, PBD are drawn to cut the other circle at C and D. Show that CD is parallel to the tangent at P. [C. U. '35 ; H. S. '63]

At P the tangent PT is drawn to the circle PAB

To prove that PT ∥ CD.

Join CD and AB.

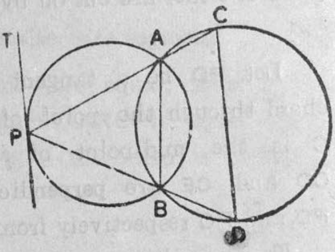


Fig. 4

Proof. ∵ PT is a tangent and PA is a chord through the point of contact,

∴ ∠APT = ∠ABP in the alternate segment

Again, ∵ ABDC is a cyclic quadrilateral,

∴ the exterior ∠ABP = the interior opposite ∠C.

∴ ∠TPC = ∠PCD, but these are alternate angles,

∴ PT ∥ CD,

Ex. 4. If two circles intersect, the angles subtended at points of intersection by a common tangent are supplementary.

Two circles intersect at A and B and PQ is a common tangent touching the circles at P and Q.

To prove that ∠PAQ + ∠PBQ = 2 right angles.

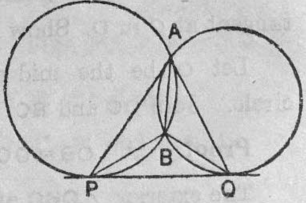


Fig. 5

Join AP, AQ, BP, BQ, AB.

Proof. ∠BPQ = ∠PAB in the alt. segment and ∠BQP = ∠BAQ in the alt. segment [∵ PQ is a tangent].

∴ the whole ∠PAQ = ∠BPQ + ∠BQP.

∴ ∠PAQ + ∠PBQ = ∠BPQ + ∠BQP + ∠PBQ = 2 rt. angles.

Ex. 5. Tangents are drawn at A, B, C to the circle circumscribing an acute-angled $\triangle ABC$ so as to form another triangle. Show that the angles of this triangle are respectively supplements of twice the opposite angles of $\triangle ABC$.

[C. U. 1939]

Let DE, DF and EF be tangents to the circum-circle of $\triangle ABC$ at A, B, C respectively and let them form the $\triangle DEF$.

To prove that $\angle D$, $\angle E$ and $\angle F$ are supplements of $2\angle C$, $2\angle B$ and $2\angle A$ respectively.

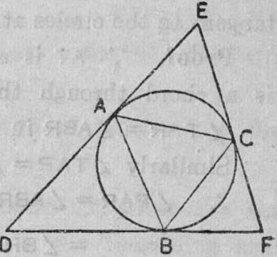


Fig. 6

Proof. $\angle DAB = \angle ACB$ in the alt. segment and $\angle DBA = \angle ACB$ in the alt. segment.

$\therefore \quad \angle DAB + \angle DBA = 2\angle ACB.$

Now, in $\triangle ABD$, $\angle D + \angle DAB + \angle DBA = 2$ rt. angles,

$\therefore \quad \angle D + 2\angle ACB = 2$ rt. angles.

$\therefore \quad \angle D$ is the supplement of $2\angle ACB$ or $2\angle C$.

Similarly, it can be proved that $\angle E =$ supplement of $2\angle B$ and $\angle F =$ supplement of $2\angle A$.

Ex. 6. A chord AB of a circle bisects the angle between the diameter through A and the perpendicular from A to the tangent at B. [C. U. '42 Addl. ; cf. D. B. 1926]

Let AB be a chord and AC, a diameter of the circle ABC. PB is a tangent to the circle at B and $AP \perp BP$.

To prove that, $\angle BAP = \angle BAC$.

Proof. Join BC. $\angle ABC$ is a right angle being in a semi-circle.

$\angle ABP = \angle ACB$ in the alt. segment.

Now, in $\triangle ABP$ and $\triangle ABC$,

$\angle APB = \angle ABC$ (being right angles).

$\angle ABP = \angle ACB$, $\therefore$ the remaining angles are equal, i. e. $\angle BAP = \angle BAC$.

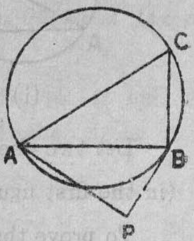


Fig. 7

Ex. 7. Two circles touch internally at A ; PQ, a chord of the outer, touches the inner circle at R. Prove that AR bisects the angle PAQ. [P. U. '33]

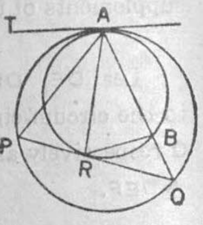


Fig. 8

Hints. Let AQ cut the inner circle at B. Join BR and draw AT a common tangent to the circles at A.

Proof. $\because$ AT is a tangent and AR is a chord through the pt. of contact,

$\therefore \angle TAR = \angle ABR$ in the alt. segment.

Similarly $\angle TAP = \angle Q$.

$\therefore \quad \angle PAR = \angle ABR - \angle Q$

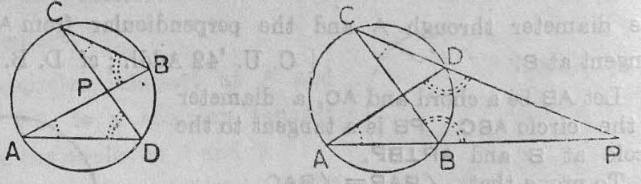
$= \angle BRQ$ ($\because$ ext. $\angle ABR = \angle Q + \angle BRQ$)

$= \angle BAR$ in the alt. segment ($\because$ PQ is a tangent).

$\therefore$ AR bisects the $\angle PAQ$.

Theorem 2

If two chords of a circle intersect, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.



(i) Fig. 9 (ii)

Let two chords AB and CD of a circle intersect at P inside (in the first figure) or outside (in the second fig.) the circle.

To prove that $AP.PB = CP.PD$.

Construction. Join AD and BC.

Proof. In $\triangle PAD$ and $\triangle PBC, \angle APD = \angle BPC$, and $\angle PAD = \angle PCB$ being on the same arc ;

$\therefore$ the remaining angles PDA and PBC are equal.

$\therefore$ the $\triangle^s PAD$, PBC are equiangular,

$$\therefore \frac{AP}{PC} = \frac{PD}{PB}, \quad \therefore \quad AP.PB = PC.PD.$$

Corollary : (i) *If a chord* AB *and a tangent* PQ *of a circle intersect at* P *outside it, then* $AP.PB = PQ^2$.

Let the chords AB and CD intersect at P outside the circle [See fig. 9 (ii)]. Join AD, BC.

Proof. In $\triangle APD$ and $\triangle PBC$, $\angle BAD = \angle BCD$ on the same arc, and $\angle P$ is common to both. $\therefore$ the remaining angles ADP, PBC are equal. $\therefore$ the triangles are equiangular.

$$\therefore \frac{AP}{PC} = \frac{PD}{PB}, \quad \therefore AP.PB = PC.PD.$$

Now, if the chord CD gradually moves away from AB towards the circumference, the points C and D will come nearer to each other. Thus when C and D will coincide, the two segments PC and PD will be equal. Suppose C and D coincide at the point Q on the arc CD. Then the st. line PQ becomes a tangent to the circle and $PQ = PC$.

$$\therefore AP.PB = PC.PD = PC.PC = PC^2 = PQ^2.$$

(2) *If from a point* P *outside a circle a secant* PBA *and a st. line* PQ *be drawn so that* $AP.BP = PQ^2$, *then* PQ *is a tangent to the circle.*

Let the secant PBA drawn from the external pt. P cut the circle at B and A and PQ drawn from P meet the circle at Q and let $PA.PB = PQ^2$.

To prove that PQ is a tangent to the circle.

Proof. If PQ is not a tangent to the circle, then it will, when produced, meet the circle at some other point, say R.

Now, since the chords AB and RQ intersect at P outside the circle, $\therefore$ AP.PB = PR.PQ ; [here give its proof as in theorem 2] but AP.PB = PQ^2 (hyp.)

$\therefore$ PR.PQ = PQ^2, $\therefore$ PR = PQ, i.e.. R coincides with Q.

$\therefore$ PQ produced does not meet the circle at any other point.

$\therefore$ PQ is a tangent to the circle.

[What will be the **Converse of Theorem 2** ? State and prove it.]

Miscellaneous Examples (2)

Ex. 1. P is any point in AB, a chord of a circle. Show how to draw a line PC from P to the circumference of the circle so that $PC^2 = PA.PB$. [C. U. '40]

Let O be the centre of a circle and P be any point on its chord AB.

Join OP and at P draw PC perpendicular to OP cutting the circle at C. Then PC is the required line.

Proof. Produce CP to cut the circle at D.

$\because$ OP$\perp$CD, $\therefore$ PC = PD.

Now, AP.BP = PC.PD [$\because$ the chords AB, CD cut at P]

$= PC.PC = PC^2$.

Ex. 2. A semi-circle is described on AB as diameter, and any two chords AC and BD are drawn intersecting at P. Show that $AB^2 = AC.AP + BD.BP$. [C. U. '47 ; D. B. '39]

Hints : Draw PM$\perp$AB. In the quad. ADPM, $\angle$D and $\angle$PMA are rt. angles, i.e., $\angle D + \angle M = 2$ rt. angles. $\therefore$ It is a cyclic quadrilateral. Now, since the chords DP, AM of the circle ADPM intersect at the external pt. B, $\therefore$ AB.BM = BD.BP$\cdots$(i)

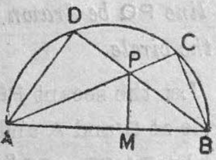


Fig. 10

Similarly, $\because$ PCBM is a cyclic quadrilateral,

$\therefore$ AB.AM = AC.AP$\cdots$(2)

Adding (1) and (2) we have

$$AB.BM + AB.AM = BD.BP + AC.AP,$$

or, $AB(BM + AM) = AC.AP + BD.BP,$

i.e., $AB.AB = AC.AP + BD.BP,$

$\therefore \quad AB^2 = AC.AP + BD.BP.$

Ex. 3. Through any point in the common chord of two intersecting circles two chords are drawn, one in each circle. Show that the four extremities of these chords are concyclic.

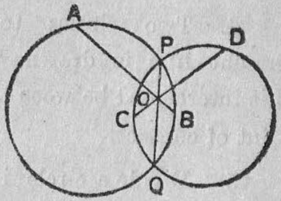


Fig. 11

Let O be a point on the common chord PQ and let two chords AOB and COD be drawn in the two circles through O.

To prove that A, C, B, D are concyclic.

Proof. The chords AB and PQ of circle PAQ intersect at O.

$\therefore$ $AO.BO = PO.QO$. Again, in the circle PDQ, $CO.DO = PO.QO$. $\therefore$ $AO.BO = CO.DO$.

$\therefore$ A, C, B and D are concylic.

Ex. 4. If two circles intersect, show that tangents drawn to them from any point in their common chord produced are equal. [C. U. 1934]

Let two circles intersect at A and B, and let TP and TR be tangents to the circles from any point T on the common chord AB produced.

To prove that $TP = TR$.

Proof. $\because$ TP is a tangent and TAB a secant of the circle ABP, $\therefore$ $TA.TB = TP^2$.

Similarly in the circle ABR, $TA.TB = TR^2$.

$\therefore \quad TP^2 = TR^2, \quad \therefore \quad TP = TR.$

Exercise 1

1. A tangent is drawn parallel to a chord, show that the intercepted arc is bisected at the point of contact.
[C. U. '45 ; D. B. '32]

2. A, B, C are points on a circle. BC produced and the tangent at A intersect at P. Prove that $\angle ACP = \angle PAB$.

3. Two circles touch each other internally at A and chords APQ, AXY are drawn. Show that $PX \parallel QY$. [C. U. '47]

4. Two circles touch each other internally and a straight line is drawn to cut them. Prove that the parts of it intercepted between the circles subtend equal angles at the point of contact. [C. U. '24]

5. Divide a circle into two segments so that the angle in one may be double of the angle in the other.

6. Two circles touch internally or externally and from the point of contact two straight lines are drawn to cut them. Prove that the lines joining the points of section are parallel.

7. ABC is a triangle right-angled at C, from C a perpendicular CD is drawn to the hypotenuse. Show that $CD^2 = AD.BD$. [C. U. '44]

8. Two straight lines AB and CD intersect at O so that $AO.BO = CO.DO$; prove that A, B, C, D are concylic.

9. Two circles intersect at A and B ; show that AB produced bisects their common tangent. [C. U. '19]

10. Two chords AB and CD of a circle intersect at O outside it. If $OB = OD$, show that $AB = CD$.

11. ABC is a triangle in which AX, BY, CZ are the perpendiculars from the vertices to the opposite sides. If the perpendiculars meet at O, prove that $AO.OX = BO.OY = CO.OZ$.
[G. U. '48]

12. Show that the rectangle contained by the segments of any chord drawn through a given point within a circle is equal to the square on half the shortest chord which may be drawn through that point. [C. U. '49]

13. If three circles intersect one another, the three common chords are either concurrent or parallel.

14. A, B, C are three points on a straight line. Find the locus of points of contact of tangents from A to the circles passing through B and C. [C. U. '49]

Construction of Tangents

Problem 1

Draw a tangent to a circle at a given point on the circumference.

Let O be the centre of the circle and P any point on the circumference.

To draw a tangent to the circle at P.

Construction. Join OP and at P draw XY perpendicular to OP.

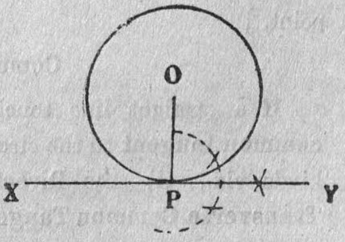


Fig. 12

Then XY is the required tangent.

Proof. ∵ XY is perpendicular to the radius OP at P,

∴ XY touches the circle at P.

Hence, XY is a tangent to the circle at P.

Problem 2

Draw a tangent to a circle from a given external point.

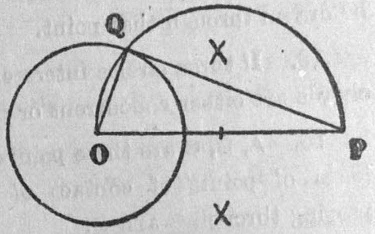


Fig. 13

Let O be the centre of the given circle and P be the given external point.

To draw a tangent to the circle from P.

Construction. Join OP. On OP as diameter draw a semi-circle cutting the given circle at Q. Join PQ.

Then PQ is the tangent required.

Proof. Join OQ. The $\angle OQP$, being in a semi-circle, is a right angle.

$\therefore$ PQ is perpendicular to the radius OQ at Q.

Hence PQ is a tangent to the circle at Q.

[**N. B.** If a complete circle be drawn on OP as diameter, it cuts the given circle at another point. Let R be the point. If PR is joined, PR is another tangent to the circle from the pt. P. Hence, **two tangents can be drawn** to a circle from an external point.]

Common Tangent

If a straight line touches two given circles, it is called a **common tangent** to the circles. The common tangent is of two kinds, viz., (i) the **Direct Common Tangent and** (ii) the **Transverse Common Tangent.**

If the two points of contact of the common tangent lie on the same side of the line of centres of the given circles, it is said to be a **direct common tangent.** But if the two points of contact lie on the opposite sides of the line of centres, the common tangent is said to be a **transverse common tangent.**

Problem 3

To draw a direct common tangent to two given circles.

Let A and B be the centres, and R and r the radii of the given circles.

Let R be greater than r.

To draw a direct common tangent to the circles.

Construction. Join AB. With centre A and radius equal to the difference of the given radii (*i.e.*, $R-r$) draw a circle.

From B, draw a tangent BC to this third circle. Join AC and produce it to meet the greater circle at P. From B, draw the radius BQ parallel to AP in the same sense. Join PQ.

Then PQ is a direct common tangent to the given circles.

Proof. $\because$ $AP=R$ and $AC=R-r$, $\therefore$ $CP=r=BQ$. Thus CP and BQ are equal and parallel.

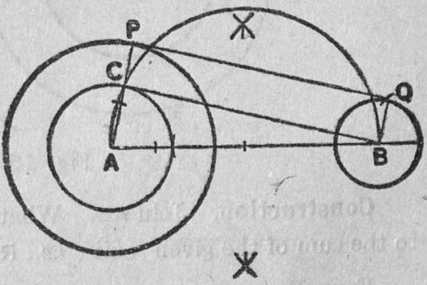


Fig. 15

$\therefore$ $BCPQ$ is a parallelogram.

Again, $\angle CPQ =$ the corresponding $\angle ACB$

$=$ one right angle (BC being a tangent),

$\therefore$ $BCPQ$ is a rectangle.

$\therefore$ $\angle CPQ$ and $\angle PQB$ are right angles. Hence PQ is a direct common tangent to the given circles.

[**N. B.** (1) Since another tangent can be drawn from B to the third circle on the opposite side of C, another direct common tangent may similarly be drawn to the given circles corresponding to PQ.

(2) Show how to draw a direct common tangent to two equal circles.]

Problem 4

To draw a transverse common tangent to two given circles.

Let A and B be the centres and R, r be the radii of the two given circles.

To draw a transverse common tangent to the circles.

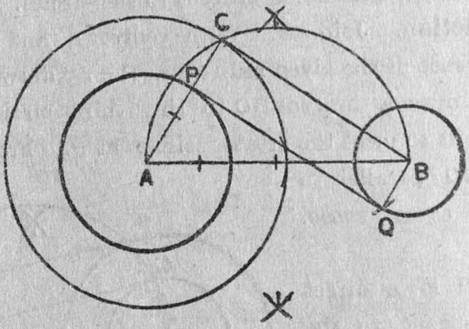


Fig. 15

Construction. Join AB. What centre A and radius equal to the sum of the given radii (*i.e.*, $R+r$) draw a circle.

From B, draw a tangent BC to this circle. Join AC cutting the given circle (A) at P. Through B draw the radius BQ parallel to AP in the opposite sense. Join PQ.

Then PQ is a transverse common tangent to the circles.

Proof. $\because$ $AC=R+r$ and $AP=R$,

$\therefore$ $PC=r=BQ$, and $PC \parallel BQ$.

$\therefore$ PCBQ is a parallelogram and its angle PCB is a right angle. $\therefore$ PCBQ is a rectangle, so the angles at P and Q are right angles. $\therefore$ PQ is a tangent to the circles. Hence PQ is a transverse common tangent to the given circles.

[**N. B.** As in Problem 3 here also another tangent can be drawn from B to the third circle and so another transverse common tangent may be drawn to the given circles.

Hence there are four common tangents to two given circles, when they are external to each other. According to the relative positions of the given circles we have 1, 2, 3, or 4 common tangents to them. There will be no common tangent, if the circles are concentric.]

Construction of regular figures in or about a given cricle.

Problem 5

To construct a regular figure of 3 sides (i) in, or (ii) about a given circle.

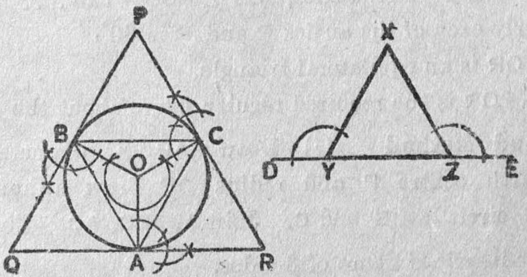


Fig. 16

[**First method**] Let O be the centre of the given circle.

To draw a regular figure of 3 sides (*i. e.*, an equilateral triangle) (*i*) in and (*ii*) about the circle.

(*i*) **Construction.** Draw any equilateral triangle XYZ and produce YZ on both sides to D and E.

Take any radius OA of the circle. At O draw ∠AOB equal to ∠DYX and ∠AOC equal to ∠XZE. Let OB and OC cut the circle at B and C. Join AB, BC, AC. Then △ABC is the required regular figure of 3 sides in the circle.

Proof. $\because$ Each angle of the equilateral $\triangle XYX$ is 60°, $\therefore$ $\angle XYD = \angle XZE = 120^\circ$. $\therefore$ at O, $\angle AOB = \angle AOC = 120^\circ$, $\therefore$ $\angle BOC = 120^\circ$. Now, $\because$ the three angles at the centre O are equal, $\therefore$ the arcs AB, BC, AC are equal.

$\therefore$ the chords AB, BC and AC that cut off the equal arcs are also equal.

$\therefore$ $\triangle ABC$ is equilateral.

Hence, $\triangle ABC$ is the required inscribed regular figure.

(*ii*) **Construction.** Draw the inscribed $\triangle ABC$ as before. Now, draw three tangents to the circle at A, B, C forming the $\triangle PQR$, then PQR is the regular figure about the circle.

Proof. $\because$ QP and QR are tangents to the circle,

$\therefore$ $\angle OBQ = \angle OAQ =$ one right angle.

$\therefore$ $\angle AOB + \angle Q = 2$ rt. angles, but $\angle AOB = 120^\circ$, $\therefore$ $\angle Q = 60^\circ$.

Similarly each of the angles P and R is 60°.

$\therefore$ PQR is an equilateral triangle.

Hence PQR is the required regular figure about the circle.

[**Second Method**] (*i*) Draw AP a diameter of the circle. With centre P and radius OP draw an arc cutting the given circle at B and C. Join AB, BC, AC. Then ABC is the regular inscribed figure of 3 sides.

Proof. Join OB, OC, BP, PC. Then $\triangle BOP$ and $\triangle COP$ are equilateral, $\therefore$ $\angle BOC = 120^\circ$.

$\therefore$ $\angle BAC$ in the circumference $= \frac{1}{2} \angle BOC$ at the centre $= 60^\circ$.

Again, $\angle BCA = \angle BPA$ in the same segment $= 60^\circ$.

$\therefore$ $\triangle ABC$ is equilateral.

(*ii*) Now, draw tangents to the circle at A, B, C.

Then the tangents will form an equilateral triangle about the circle.

[**Third Method**] Let OA be any radius. Draw the chords AX, XC equal to OA. Join AC and draw the chord CB equal to AC. Then $\triangle ABC$ is equilateral [proof is easy].

Problem 6

To construct a regular figure of 4 sides (i) in, or (ii) about a given circle.

[You know that a regular figure of 4 sides is a square.]

Let O be the centre of the given circle.

To draw a regular quadrilateral (*i*) in and (*ii*) about the circle.

Construction. Draw two diameters AB and CD at right angles to each other.

(*i*) Join AC, CB, BD, DA. Then ACBD is the required inscribed regular quadrilateral.

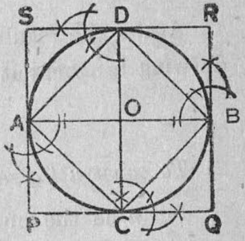


Fig. 17

Proof. The angles at O are right angles.

∵ the chords AC, BC, BD and AD subtend equal angles at the centre, ∴ they are equal.

Again, ∠CAD, ∠ACB, ∠CBD, ∠ADB are right angles, being in semi-circles.

∴ ACBD is a regular figure of 4 sides, *i. e.*, a square in the given circle.

(ii) Draw four tangents to the circle at A, C, B and D. Let them intersect at P, Q, R, S and form the quadrilateral PQRS. Then PQRS is the required regular quadrilateral about the circle.

Proof. ∵ AB and SR are perpendicular to the same st. line OD, ∴ SR ∥ AB. Similarly AS ∥ BR.

∴ ABRS is a parallelogram ; but ∠BAS is a right angle,

∴ ABRS is a rectangle. Similarly ABQP is a rectangle.

∴ SR = AB = PQ. Similarly PS = CD = RQ.

But AB = CD, ∴ PS = PQ.

∴ PQRS is a regular four-sided figure or a square about the given circle.

Corollary. To construct a regular octagon in and about the given circle, bisect the four angles between the two diameters AB, CD. Let the bisectors cut the circle at E, F, G, H. Join the successive points on the circumference. The figure thus formed is a regular octagon in the circle.

[Proof is easy].

Again, the regular octagon about the circle is obtained by drawing tangents at the eight points.

Problem 7

To construct a regular polygon in a circle.

Let n be the number of sides of the polygon and O the centre of the given circle.

At O draw $\angle AOB = \frac{360^\circ}{n}$. Let OA and OB cut the circle at A and B respectively. Join AB.

Draw the chords BC, CD, etc. equal to the chord AB. Thus the inscribed regular polygon of n sides (*i.e.*, the regular n-gon) is formed. [Proof is easy]

Again, the regular n-gon about the circle is obtained by drawing tangents to the circle at the points of section, A, B, C, D, etc.

Regular figures of any number of sides (3, 4, etc,,) may also be drawn by this method.

Medial Section

When a straight line is divided into two parts so that the rectangle contained by the whole line and one part is equal to the square on the other part, it is said to be divided in **medial section.**

The point of section is called the **point of medial section.**

(*a*) *To divide a given straight line in medial section.*

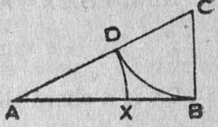


Fig. 18

Let AB be the given straight line to be divided in medial section.

Construction. Draw BC perpendicular to AB and make $BC = \frac{1}{2}AB$. Join AC and from CA cut off CD equal to BC. From AB cut off AX equal to AD.

Then AB is divided at X in medial section.

Proof. $\because$ $\angle B$ is a right angle,

$$\begin{aligned}\therefore\quad AB^2 &= AC^2 - BC^2 = AC^2 - CD^2 = (AC+CD)(AC-CD)\\ &= (AD+CD+CD).AD = (AD+2CD).AD\\ &= (AD+2BC).AD = (AD+AB)AD \qquad [\because BC = \tfrac{1}{2}AB]\\ &= AD^2 + AB.AD,\end{aligned}$$

$\therefore\quad AB^2 - AB.AD = AD^2$, or, $AB(AB-AD) = AD^2$,

or, $AB(AB-AX) = AX^2$ $[\because AD = AX]$

or. $AB.BX = AX^2$.

Hence AB is divided in medial section at X.

[**N. B.** Here AB is divided **internally** in medial section. To divide it **externally** in medial section, from AC produced cut off CD equal to BC, and produce BA to E so that $AE = AD$. Then AB is divided externally in medial section at E.]

(b) *To construct an isosceles triangle having each of the angles at the base double of the vertical angle.* [C. U. '20, '37]

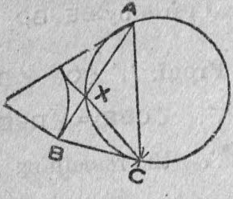


Fig. 19

Construction : Take any st. line AB and divide it in medial section at X so that $AB.BX = AX^2$. With centres B and X and radius AX draw two arcs cutting each other at C. Join AC, BC and CX. Then ABC is the triangle required.

Proof. Draw the circum-circle of $\triangle AXC$,

Now, $AB.BX = AX^2 = BC^2$ ($\because$ $BC = AX$),

$\therefore$ BC is a tangent to the circle AXC at C.

CX is a chord through the point of contact.

$\therefore$ $\angle BCX = \angle CAX$ in the alternate segment.

$\because$ $AX = CX$ (by construction). $\therefore$ $\angle ACX = \angle CAX$.

$\therefore$ the whole $\angle ACB = 2\ \angle CAX = 2\ \angle A$.

Again, $\angle BXC = \angle CAX + \angle ACX = 2\ \angle A$;

but $\angle CBX = \angle BXC$ ($\because$ $BC = AX = CX$), $\therefore$ $\angle B = 2\angle A$.

$\therefore$ $\angle B = \angle ACB$. $\therefore$ $\triangle ABC$ is an isosceles triangle and its $\angle B = \angle C = 2\ \angle A$.

(c) *Divide a right angle into five equal parts.*

Let ABC be a right angle to be divided into five equal parts.

With the side BC draw an isosceles triangle BCD so that each of the angles C and D is double of the vertical $\angle CBD$.

Now, bisect the $\angle CBD$ by BE. At B draw $\angle DBF$ and $\angle FBG$ equal to $\angle CBE$. Then the right angle is divided into five equal parts by BE, BD, BF and BG.

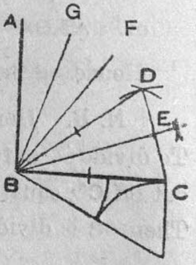


Fig. 20

Proof. $\because \angle C = \angle D = 2\angle CBD$, $\therefore$ $\angle C = 72^\circ$, $\therefore$ $\angle CBD = 36^\circ$,

$\therefore$ $\angle CBE = \angle DBE = 18^\circ$. $\therefore$ $\angle DBF = \angle FBG = \angle CBE = 18^\circ$.

$\therefore$ the remaining $\angle ABG = 90^\circ - 18^\circ \times 4 = 18^\circ$.

Thus the right angle is divided into five equal parts.

Problem 8

To construct a regular pentagon (*i*) *in, or* (*ii*) *about a given circle.* [C. U. '15, '34, '17, '47]

Let O be the centre of the given circle in which a regular pentagon is to be inscribed.

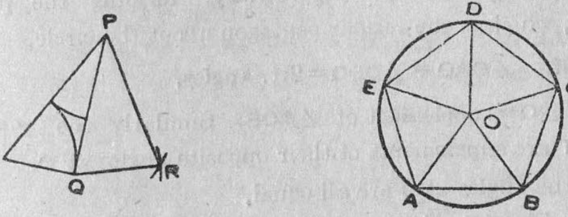


Fig. 21

(i) **Construction.** Draw an isosceles triangle PQR so that each of the angles Q and R is double of $\angle P$. Draw any radius OA and at O draw $\angle AOB = \angle Q$. Let OB cut the circle at B. Join AB. Draw the chords BC, CD, DE each equal to AB and join AE.

Then ABCDE is the inscribed regular pentagon.

Proof. $\because$ $\angle Q = 2\angle P$ and $\angle R = 2\angle P$,

$\therefore$ $\angle P + \angle Q + \angle R = 5\angle P$. $\therefore$ $5\angle P = 180^\circ$,

$\therefore$ $\angle P = 36^\circ$, $\therefore$ $\angle Q = 72^\circ$.

Now, $\because$ AB, BC, CD, DE are equal chords,

$\therefore$ the angles subtended by them at the centre are equal, *i.e.*, $\angle AOB = \angle BOC = \angle COD = \angle DOE = 72^\circ$.

$\therefore$ the sum of the four angles $= 72^\circ \times 4 = 288^\circ$, but the angles at $O = 360^\circ$, $\therefore$ $\angle AOE = 360^\circ - 288^\circ = 72^\circ$,

$\therefore$ the chord AE is equal to the other chords.

Again, $\because$ $\triangle AOB$ is isosceles, $\therefore$ $\angle OAB = \angle OBA$, and $\because$ $\angle AOB = 72^\circ$, $\therefore$ $\angle OAB + \angle OBA = 180^\circ - 72^\circ = 108^\circ$.

∴ ∠OAB = ∠OBA = 54°. Thus each base angle of the triangles in the figure is 54°.

∴ each angle of the pentagon = 54° × 2 = 108°.

∴ ABCDE is a regular pentagon inscribed in the circle.

(ii) Draw PQ, QR, RS, ST and PT tangents to the circle at A, B, C, D and E respectively, forming the pentagon PQRST, which is the regular pentagon about the circle.

Proof. ∠OAQ + ∠OBQ = 2rt. angles,

∴ ∠Q = supplement of ∠AOB. Similarly the ∠P, ∠R, ∠S, ∠T are supplements of their opposite angles at O.

But the angles at O are all equal,

∴ ∠P, ∠Q, ∠R, ∠S, ∠T are equal.

Again, join PO, QO. △ AOQ, BOQ are congruent.

∴ ∠AOQ = ∠BOQ = $\frac{1}{2}$∠AOB. Similarly ∠AOP = $\frac{1}{2}$ ∠AOE.

∴ ∠AOQ = ∠AOP. ∴ △AOP and △AOQ are congruent,

∴ AQ = AP, ∴ PQ = 2AQ. Similarly QR = 2BQ, but AQ = BQ, ∴ PQ = QR. ∴ PQRST is equilateral.

∴ PQRST is a regular pentagon about the given circle.

Problem 9

To construct a regular hexagon (*a figure of 6 sides*) (*i*) *in, or* (*ii*) *about a given circle.*

Let O be the centre of the given circle. To construct a regular hexagon (*i*) in and (ii) about the circle.

(i) **Construction.** Draw any radius OA and draw the chords AB, BC, CD, DE and EF equal to OA. Join AF.

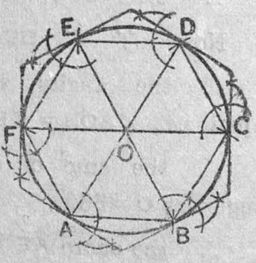


Fig. 22

Then ABCDEF is the regular hexagon in the given circle.

Proof. Join OA, OB, OC, etc. Now, since the five chords AB, BC, CD, DE and EF are equal, they subtend equal angles at the centre O.

Again, ∵ $\triangle^s$ AOB, BOC, etc., are equilateral, each of the five angles AOB, BOC, COD, DOE, EOF at the centre is 60°. ∴ the remaining angle AOF is also 60° (∵ the angles at O $=360°$). ∴ $\triangle$AOF is equilateral.

∴ The hexagon is equilateral and each of its angles ABC, BCD, etc., being $60° \times 2$ or 120°, all its angles are equal.

Hence, ABCDEF is a regular hexagon inscribed in the given circle.

(ii) **Construction.** Draw six tangents to the given circle at A, B, C, D, E, F. They intersect and form a regular hexagon about the given circle. [Proof is similar to that of Problem 8.]

Problem 10

To construct a square equal in area to a given rectangle.

[C. U. ; D. B. ; G. U.]

Let ABCD be the given rectangle. To construct a square equal in area to the rectangle ABCD.

Construction: Produce AD to E so that $DE=DC$. Draw a semicircle on AE as diameter. Produce CD to meet the semi-circle at F. Draw a square on DF, which is the square required. [You should draw the square in the figure]

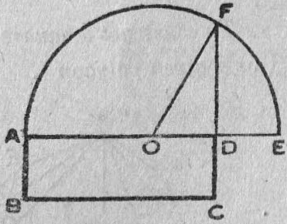


Fig. 23

Proof. Let O be the mid point of the diameter AE.

Join OF. Then $OA=OF=OE$. ∵ $\angle D$ is a right angle,

$$\therefore \quad DF^2=OF^2-OD^2=OE^2-OD^2=(OE+OD)(OE-OD)$$

$$=(AO+OD).DE=AD.DE=AD.DC.$$

∴ The square on $DF=$ the rectangle ABCD.

Corollary :—(1) To construct a rectangle equal to a given square.

Hints : [Draw a fig.] Let a be a side of the square. Take a straight line AB greater in length than $2a$ and draw a semi-circle on it as diameter. At A draw $AC \perp AB$ so that $AC = a$.

Draw $CE \parallel AB$, cutting the semi-circle at D and E. Draw $DF \perp AB$. Then the rectangle contained by AF, BF is the required rectangle.

The $\angle ADB$ in the semi-circle is a right angle.

$\because$ DF is perpendicular from D to the hypotenuse AB,

$\therefore$ $a^2 = AC^2 = DF^2 = AF.BF$.

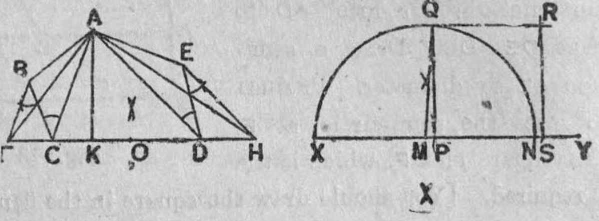
(2) To divide a given st. line so that the rectangle contained by the segments may be equal to a given square.

[Hints : Let AB be the given st. line and l be a side of the given square. Now, proceed as in *cor.* (1) above.]

N.B. See Appendix for other cases.

Problem 11

To construct a square equal in area to a given rectilineal figure (or a given polygon).



(Fig. 23A)

Let ABCDE be the given rectilineal figure.

To construct a square equal to it in area.

Construction. Join AC, AD, and draw $BF \parallel AC$ and $EH \parallel AD$. Let BF, EH cut CD produced at F, H respectively.

Join AF and AH and draw $AK \perp FH$. Then AK is the altitude of $\triangle AFH$. Bisect FH at O.

From any st. line XY cut off $XP = FO$ and $PN = AK$.

Draw a semi-circle on XN as diameter and let M be the middle point of XN.

Draw $PQ \perp XY$ so that PQ meets the semi-cirele at Q. On PQ draw a square $PQRS$, which is the required square.

Proef. $\because$ $\triangle^s ACF$, ABC stand on the same base AC and between the same parallels AC and BF, $\therefore$ $\triangle ABC = \triangle ACF$.

Similarly $\triangle ADE = \triangle ADH$.

$\therefore$ $\triangle ABC + \triangle ADC + \triangle ADE = \triangle ACF + \triangle ADC + \triangle ADH$,

$\therefore$ the fig. $ABCDE = \triangle AFH = \frac{1}{2} FH.AK = FO.AK$.

$$\begin{aligned} \text{Again, } PQ^2 &= QM^2 - PM^2 \quad (\because \angle P \text{ is a rt. angle}) \\ &= MX^2 - PM^2 \quad (\because MX = QM = \text{radius}) \\ &= (MX + PM)(MX - PM) = (MX + PM)(MN - PM) \\ &= PX.PN = FO.AK = \text{fig. } ABCDE. \end{aligned}$$

$\therefore$ The square on PQ = the rectilineal fig. $ABCDE$.

Problem 12

To find the mean proportional between two given straight lines.

Let a and b be the given st. lines. To find a mean proportional between a and b.

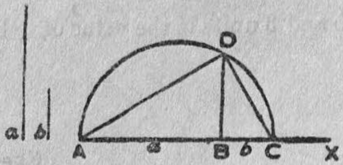


Fig 24

Construction. Take any st. line AX and from it cut off $AB = a$ and $BC = b$. Draw a semi-circle on AC as diameter and draw $BD \perp AC$ so that BD meets the semi-circle at D. Then BD is the mean proportional required.

Proof. Join AD, DC. $\angle ADC$ in the semi-circle is a right angle.

$\because$ BD is perpendicular to the hypotenuse AC from the right angle, $\because$ $\triangle$ ABD, DBC are similar.

$\therefore \dfrac{AB}{BD}=\dfrac{BD}{BC}$, $\therefore \dfrac{a}{BD}=\dfrac{BD}{b}$,

$\therefore$ BD is the mean proportional between a and b.

[**N. B.** In fig. 24, let AC and AB be the given st. lines. Let them be placed one upon the other so that the two extremities A coincide. In this case AD is the mean proportional between AC and AB, and CD is the mean proportional between CA and CB.

12. (*a*) Find geometrically the value of $\sqrt{5}$.

[In fig. 24] Take a st. liue $AC=5$ units of length.

From AC cut off $AB=1$ unit of length.

Draw AD the mean proportional between AB and AC.

Now, $\because$ AD is the mean proportional between AB and AC,

$\therefore AD^2=AB.AC=5.1$ sq. units $=5$ square units.

$\therefore AD=\sqrt{5}$ units of length.

$\therefore$ AD is the geometric measure of $\sqrt{5}$.

[**N. B.** $\sqrt{15}=\sqrt{5\times 3}$, so the mean proportional between 5 and 3 units is the value of $\sqrt{15}$. Similarly $\sqrt{34}=\sqrt{6\cdot 8\times 5}$, etc.]

Exercise 2

1. Find the mean proportional between 3 cm. and 4 cm.
2. Find geometrically the values of $\sqrt{35}$ and $\sqrt{26}$.
3. Draw a regular decagon in a given circle.

4 Draw a regular polygon of 12 sides in or about a given circle.

5. Describe a circle in, or about, a regular polygon.

6. Inscribe a square in a given circle.

7. Inscribe a regular octagon in a circle of radius 5 cm.
[C. U. '35]

8. Inscribe in a circle of radius 2 cm. a square and find its side by measurement and calculation. [C. U. '51]

[Ans. = 2·82 cm.]

9. Construct a regular hexagon circumscribing a circle of radius 1·5 inches. Measure a side of this hexagon.
[Pat. U. '50]

10. About a circle of radius $1''$, describe an equilateral triangle. Draw a square whose area is equal to that of the triangle. Measure the sides of the square and the triangle.
[U. U. '51]

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SOLID GEOMETRY

Solid Geometry deals with the properties of points, lines, planes and solids in a three dimensional space.

1. **Dimension.** The length, breadth and thickness are each called a **dimension** of a body.

2. **Point.** A point has position only but no dimension, that is, it has neither length nor breadth nor thickness.

3. **A line** has length but no breadth and thickness. So it has one dimension only.

[**N. B.** You know that a line is traced out by the motion of a point. Lines, therefore, meet at a point.]

A straight line is the shortest distance (or line) between two given points on a plane.

4. **Surface** : A surface has length and breadth but no thickness. So it has two dimensions.

[**N. B.** It is evident that a surface is generated by the motion of a line. Surfaces, therefore, meet in lines. A surface is bounded by lines. A line intersects a surface in points.]

5. **Solid.** A solid has three dimensions, that is, length, breadth and thickness.

[**N. B.** A solid is generated by the motion of a surface, so a solid is bounded by surfaces.]

6. **A plane** or **a plane surface** is a surface such that the straight line joining any two points in it lies wholly in it.

7. Lines or points which lie in the same plane (or through which a plane may be made to pass) are said to be **coplanar.**

8. Two st. lines are said to be **parallel** when lying in the same plane they do not meet though indefinitely produced both ways.

[**N. B.** Two coplanar st. lines either meet at a point or are parallel.]

9. Two straight lines through which a plane cannot be made to pass are said to be **skew** or non-coplanar.

Two skew st. lines never meet however far they may be produced, but they are not parallel as they are not coplanar.

Two pencils, one placed on the other, may be an instance of skew lines.

10. **The angle between two skew st. lines** is the angle between one of them and the straight line drawn parallel to the other through any point in the first.

Let AB, CD be two skew st. lines and let PQ be drawn parallel to CD from any point P in AB. Now, the angle between AB and PQ is the angle between the skew st. lines AB, CD.

11. A straight line is said to be **parallel to a plane**, if it does not meet the plane though indefinitely produced.

12. **Parallel planes.** Two or more planes are said to be parallel, if they do not meet even when they are extended indefinitely.

[**N. B.** It is to be noted that in Solid Geometry, a st. line is supposed to be of infinite length and a plane of infinite extent.]

13. A straight line is said to be **perpendicular** or **normal** to a plane if it is perpendicular to every straight line drawn in the plane through the point at which it meets the palne.

The st. line PO meets the plane MN at O. If PO is perpendicular to all st. lines, such as AO, BO, CO, etc., drawn in the plane through O, then PO is perpendicular to the plane MN.

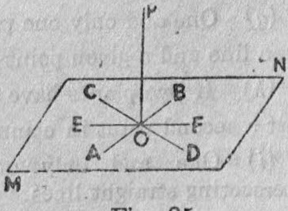


Fig. 25

14. Vertical line or plane : A straight line or a plane is said to be **vertical** when the st. line or the plane is parallel to the direction of the plumb line hanging freely at rest.

15. Horizontal line or plane : A straight line or a plane is said to be horizontal when it (st. line or plane) is perpendicular to a vertical line.

16. Skew Quadrilateral : A quadrilateral having two of its adjacent sides in one plane and two other adjacent sides in another plane is said to be a skew quadrilateral.

Axioms

17. The following will be considered as self-evident truths or **axioms** in Solid Geometry :

(*a*) Two straight lines can meet at but one point.

(*b*) A straight line can meet a plane at one point only.

(*c*) Only one straight line can be drawn joining two given points.

(*d*) A straight line joining any two points on a plane must lie wholly in the plane even if it be produced indefinitely in either direction.

(*e*) If a plane rotates about any straight line in it, it may be made to pass through any point in space.

(*f*) An infinite number of planes may be made to pass through any straight line or through any two points.

(*g*) One and only one plane may be made to pass through a given line and a given point outside it.

(*h*) If two planes have one point in common, they have at least a second point in common.

(*i*) One and only one plane may pass through two intersecting straight lines.

[**N. B.** (*a*) A straight line may be related to a plane in the following three ways :

(i) If a st. line is parallel to a plane, there is no common point between them ;

(ii) If a st. line intersects a plane, it has only one point in common with the plane ;

(iii) If a st. line and a plane have two points in common, then the line lies wholly in the plane.

(*b*) Two coplanar st. lines must either intersect or be parallel.

(*c*) A plane is determined uniquely, if

(i) three non-collinear points be given ;

(Thus any three st. lines forming a triangle must lie in a plane.)

(ii) a st. line and a point outside it be given ;

(iii) two parallel st. lines be given ;

(iv) two intersecting st. lines be given.

Miscellaneous Examples (3)

Ex. 1. If a st. line intersects two parallel st. lines, then the three st. lines are coplanar.

Let the st. line NM cut the parallel st. lines AB, CD at E and F respectively. To prove that AB, CD, MN are coplanar.

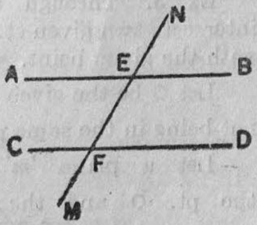


Fig. 26

Proof. ∵ AB ∥ CD, ∴ they are coplanar.

Now, ∵ E and F are points on AB and CD respectively, ∴ E and F must lie in the plane of AB, CD.

∴ the st. line EF or MN must lie in that plane.

∴ AB, CD and MN are coplanar.

Ex. 2. Show that if three or more parallel st. lines intersect a given st. line, they are coplanar. [C. U. '15, '21]

Let the parallel st. lines AB, CD, EF cut the st. line MN at P, Q, R respectively. It is required to prove that AB, CD, EF are coplanar.

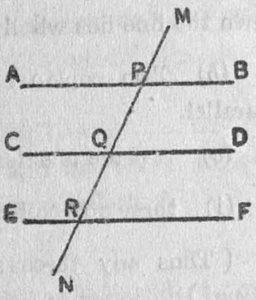


Fig. 27

Proof. $\because$ AB $\parallel$ CD, $\therefore$ a plane may be made to pass through them. Let '*p*' be that plane. Now since the pts. P and Q are in the plane '*p*', the st. line PQ, *i.e.*, MN must lie in that plane.

$\therefore$ the pt. R is also in plane '*p*'.

Now, $\because$ AB $\parallel$ EF, $\therefore$ they are coplanar. Suppose they lie in another plane '*q*'. So the point R lies in the plane '*q*'. But two planes *p*, *q* cannot pass through the st. line AB and the pt. R outside it.

$\therefore$ *p* and *q* are one and the same plane.

$\therefore$ EF lies in the plane '*p*'. $\therefore$ AB, CD, EF are coplanar,

Similarly it can be proved that any st. line parallel to AB, CD, EF and cutting MN is coplanar with AB, CD, EF.

Ex. 3. Through a given point draw a st. line which intersects two given st. lines not lying in one and the same plane with the given point. [C. U. '12]

Let O be the given point and AB, CD be the given st. lines not being in the same plane.

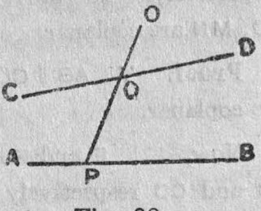


Fig. 28

Let a plane '*m*' pass through the pt. O and the st. line AB. Suppose '*m*'. intersects CD at Q. Then OQ must lie in the plane '*m*', *i.e.*, OQ and AB are coplanar.

$\therefore$ they must intersect at a point

Hence the st. line OQ is the required st. line.

[**N. B.** If OQ $\parallel$ AB, they will not meet and then no such st. line can be drawn.]

Ex. 4. Prove that the common sections of any three planes (non-collinear) meet at a point. [C. U. '11]

Let m, n, p be the three planes.

To prove that they meet at a point.

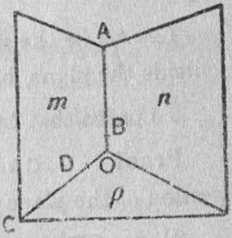


Fig. 29

Proof. The common section of two planes is a st. line. Here let the common section of m and n be AB and that of m and p be CD. Now if AB is not parallel to p, AB and CD must intersect at a pt. Let O be that point.

Hence, the three planes meet at O.

[**Instance :** The floor and two adjacent walls of a room meet at a point.]

Ex. 5. Any three straight lines forming a triangle are coplanar.

Let the three st. lines AB, BC and CA form the △ABC.

∵ One and only one plane may be made to pass through two intersecting st. lines AB and BC, ∴ AB, BC are coplanar. Hence the st. line joining A, C must lie in the same plane with AB, BC. ∴ The sides AB, BC, CA of △ABC are coplanar.

Ex. 6. Draw a st. line to cut three given non-coplanar st. lines. [C. U. '13]

Let AB, CD and PQ be three non-coplanar st. lines. To draw a st. line cutting them.

Draw a plane through AB. Now rotate the plane about AB so that it intersects CD and PQ at X and Y respectively and XY is not parallel to AB. ∵ X, Y and AB lie in the same plane, ∴ the st. line XY must intersect AB at some point. Hence XY is the required st. line.

Ex. 7. If a st. line outside a given plane is parallel to any st. line drawn in the plane, it is parallel to the plane itself. [C. U. '31]

Let AB be the st. line in the plane PQ and let CD, a st. line outside the plane, be parallel to AB.

To prove that CD is parallel to the plane PQ.

Proof. $\because$ CD $\parallel$ AB, $\therefore$ they are coplanar. $\therefore$ the common section of the plane PQ and the other plane is AB.

Hence CD produced can never intersect PQ-plane, because if CD intersect the plane PQ, the pt. of intersection must lie on AB which is impossible ($\because$ AB $\parallel$ CD).

$\therefore$ CD is parallel to the plane PQ.

Ex. 8. If a st. line is parallel to each of two planes, prove that it is parallel to their line of intersection. [C. U. '34]

Let XY be the line of intersection of the two planes M and N and let PQ be parallel to each of the two planes. To prove that PQ $\parallel$ XY.

Suppose the plane drawn through PQ parallel to the plane M intersects the plane N in the st. line AB. Then AB $\parallel$ PQ. $\therefore$ the two planes M and PA are parallel and the plane N intersects them in XY and AB respectively. $\therefore$ XY $\parallel$ AB.

Now, $\because$ XY and PQ are each parallel to AB, $\therefore$ PQ $\parallel$ XY.

Ex. 9. AB and CD are two intersecting st. lines; EF is another st. line parallel to CD and meeting AB at some point. Show that the three lines lie in one plane. [C. U. '56]

Let AB, CD intersect at O and AB, EF intersect at P.

Now, $\because$ EF $\parallel$ CD, $\therefore$ CD, EF are coplanar. Let them lie in the plane M. Since the pts. P and O lie in M-plane, the st. line PO, *i.e.*, AB also lies in that plane.

$\therefore$ AB, CD, EF lie in one plane.

Ex. 10. Two planes drawn one through each of two parallel st. lines cut one another in a st. line parallel to them.

[C. U. '22]

Let the two planes M and N, drawn respectively through the two parallel st. lines AB and CD, cut one another in the st. line PQ. To prove that PQ is parallel to AB and CD.

Proof. $\because$ AB $\parallel$ CD and CD lies in N-plane,

$\therefore$ AB is parallel to N-plane. Again, $\because$ AB lies in M-plane and the planes M, N have intersected in PQ,

$\therefore$ AB $\parallel$ PQ, but AB $\parallel$ CD, $\therefore$ PQ $\parallel$ CD.

$\therefore$ PQ is parallel to AB and CD.

Exercise 3

1. Show that any number of st. lines passing through a given point and intersecting a given st. line are coplanar.

[C. U. '54]

2. Straight lines which intersect one another mutually are coplanar.

3. If a plane cut two parallel planes, then the lines of section are parallel.

4. Show that through a given point P, a plane may be constructed parallel to each of two st. lines AB and CD, which do not lie in the same plane. [C. U. '31]

5. The sides of a quadrilateral, of which the diagonals cut one another, lie in the same plane.

6. Draw a plane cutting two sides of a triangle proportionally.

7. The sides of a parallelogram are coplanar.

Theorem 1

If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it is perpendicular to the plane in which they lie.

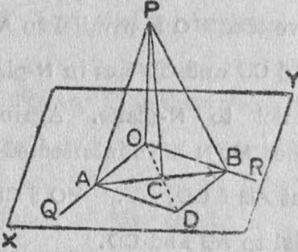


Fig. 30

Let the straight line OP be perpendicular to each of the intersecting st. lines OQ and OR at their point of intersection O and let XY be the plane in which OQ and OR lie.

It is required to prove that OP is perpendicular to the plane XY.

Cons. Through O draw *any* st. line OD in the plane XY and complete the parallelogram OADB in the plane. Join AB cutting OD at C. Join PA, PB, PC.

Proof. The diagonals of the parallelogrm OADB bisect each other at C, $\therefore$ PC is a median of the $\triangle APB$.

$\therefore \quad AP^2+PB^2=2AC^2+2PC^2 \cdots (1).$

Similarly, OC is the median of the $\triangle OAB$,

$\therefore \quad OA^2+OB^2=2AC^2+2OC^2 \cdots (2)$

$\therefore$ Subtracting (2) from (1) we have

$$(AP^2-OA^2)+(PB^2-OB^2)=2(PC^2-OC^2) \ldots (3).$$

Again, $\because$ OP is perpendicular to OA and OB,

$$\therefore \quad \left.\begin{array}{l} PA^2=OA^2+OP^2 \\ \text{and} \quad PB^2=OB^2+OP^2 \end{array}\right\}$$

$\therefore \quad PA^2-OA^2=OP^2$ and $PB^2-OB^2=OP^2 \cdots (4)$

Hence from (3) and (4) we have

$OP^2 + OP^2 = 2(PC^2 - OC^2)$, or, $2OP^2 = 2(PC^2 - OC^2)$,

or, $OP^2 = PC^2 - OC^2$, $\therefore$ $PC^2 = OP^2 + OC^2$,

$\therefore$ $\angle POC$ is a right angle.

$\therefore$ OP is perpendicular to OD, *any st. line* drawn in the plane XY through O.

$\therefore$ OP is perpendicular to the plane XY in which OA and OB lie.

Miscellaneous Examples (4)

Ex. 1. From O, the centre of a circle, a perpendicular OA is erected to the plane. Show that all points on the circumference are equidistant from A. [C. U.]

Let O be the centre of the circle and OA be perpendicular to the plane of the circle. Let B, C be *any* two points in the circumference of the circle. Join AB and AC. It is required to prove that AB = AC. Join OB and OC.

Proof. $\because$ OB and OC lie in the plane of the circle and OA is perpendicular to the plane,

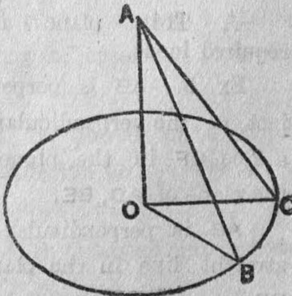


Fig. 31

$\therefore$ OA is perpendicular to OB and OC. $\therefore$ $\angle AOB$ and $\angle AOC$ are right angles. Now in the right-angled $\triangle^s$ AOB and AOC, OB = OC (being radii of a circle) and OA is common to both, $\therefore$ the $\triangle^s$ are congruent.

$\therefore$ AB = AC.

Ex. 2. Find the locus of a point in space equidistant from two given points. [C.U. '39, '47]

Let A and B be the two given points.

Join AB. Bisect AB at O and through O draw a plane perpendicular to AB. Then this plane is the required locus.

Proof. Let P be any point in this plane.

Join PO, PA, PB. $\because$ OP is a st. line lying in this plane and OB is perpendicular to the plane, $\therefore$ OB$\perp$OP.

Now, in $\triangle^s$ AOP and BOP, AO = BO, PO is common to both and $\angle$AOP = $\angle$BOP (being rt. angles).

$\therefore$ The $\triangle^s$ are congruent.

$\therefore$ AP = BP. Similarly it can be proved that any point in this plane is equidistant from A and B.

$\therefore$ This plane is the required locus.

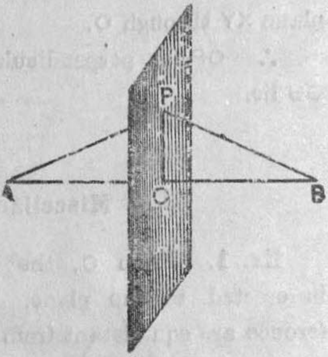


Fig. 32

Ex. 3. AB is perpendicular to a plane and if from B, the foot of the perpendicular, the line BE is drawn perpendicular to a line CF in the plane, show that CE is perpendicular to the plane of AD, BE. [C. U. '50]

AB is perpendicular to the plane XY at B and CF is any straight line in the plane XY. BE is drawn perp. to CF. AE is joined. To prove that the st. line CE is perpendicular to the plane of AE and BE.

Join AC.

Proof. AB is perp. to BE and BC, as it is perp. to the plane XY. Now from the right-angled triangles ABE and BEC we have $AE^2 = AB^2 + BE^2$ and $EC^2 = BC^2 - BE^2$,

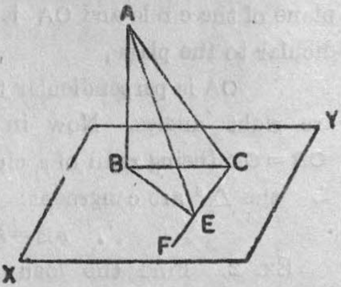


Fig. 33

$$\therefore \quad AE^2+EC^2=AB^2+BE^2+BC^2-BE^2=AB^2+BC^2$$
$$=AC^2 \; [\because \; \angle ABC \text{ is a rt. angle}]$$

$\therefore$ $\angle AEC$ is a rt. angle. $\therefore$ $CE \perp AE$; but $CE \perp BE$, $\therefore$ CE is perpendicular to the plane in which AE and BE lie.

Ex. 4. One and only one perpendicular can be drawn to a plane through a given point outside the plane.

Let O be a point outside the plane M. It is required to prove that one and only one perpendicular can be drawn from O to the the plane-M.

Proof. If possible let OA and OB be two perpendiculars drawn from O to the plane. Now, suppose the plane-N is drawn through OA and OB, and let it cut the M-plane in the straight line CD.

$\because$ OA, OB are both perp. to the plane-M and CD meets them in that plane, $\therefore$ OA, OB are both perpendicular to CD, but OA and OB lying in the same plane cannot both be perpendicular to CD. Hence only one perpendicular can be drawn from O to the plane-M.

Ex. 5. Prove that a point can be found in a plane equidistant from three given points outside the plane. State the exceptional case, if any. [C. U. '36]

Let m be the given plane and A, B, C be the three points outside the plane.

Draw the plane p bisecting AB at right angles. Then every point in this plane is equidistant from A and B. Let this plane intersect the plane m in the line ab. Then every point on ab is equidistant from A, B. Again draw the plane n bisecting BC at right angles. Then every point in plane n is equidistant from B and C. Let this plane intersect the plane m in the line cd. Then every point in cd is equidistant from B, C.

$\therefore$ every point on the line of intersection of the planes p and n is equidistant from A, B and C,

∴ the point of intersection of *ab* and *cd* is equidistant from A, B and C. Let *ab* and *cd* intersect at X.

Then the point X is equidistant from A, B and C.

Hence X is the required point.

If the line of intersection of the planes *p* and *n* be parallel to the plane *m* (*i. e.*, if A, B, C lie on a st. line parallel to the plane *m*), no such point can be found.

Exercise 4

1. Draw a plane perpendicular to a given straight line through a given paint in it.

2. Prove that one and only one perpendicular can be drawn to a plane through a given point in it.

3. Prove that all points in space equidistant from two given points lie in a plane. [C. U. '39]

4. Find the locus of a point in space equidistant from three given non-collinear points. [C.U. '51]

Theorem 2

All straight lines drawn perpendicular to a given straight line at a given point in it are coplanar.

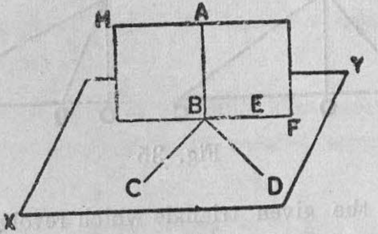


Fig. 34

Let the straight lines BC, BD, BE be each perpendicular to the st. line AB at B.

It is required to prove that BC, BD, BE are coplanar.

Cons. Only one plane can be drawn through two intersecting st. lines. Let XY be the plane drawn through BC and BD and HF be the plane drawn through the AB and BE. Let the two planes intersect in the straight line BF.

Proof. ∵ AB is perpendicular to BC and BD at B,

∴ AB is perp. to the plane XY. ∴ AB is perp. to BF lying in XY ; but AB⊥BF, ∴ of the three st. lines AB, BE, BF lying in the same plane, BE and BF are perp. to AB at the same point B, which is impossible unless BE and BF coincide.

∴ BE must lie in the plane XY

∴ BC, BD and BE are coplanar.

Similarly, any other st. line drawn perpendicular to AB at B lies in the plane XY.

Miscellaneous Examples (5)

Ex. 1. If a triangle revolves about its base, show that the vertex describes a circle. [C. Pre-U. '54, C. U. '19]

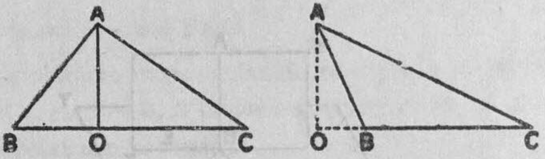


Fig. 35

Let ABC be the given triangle which revolves about the base BC as the axis. Two prove that the vertex A will describe a circle. Draw AO perpendicular to BC.

Proof. ∵ $OA \perp BC$, ∴ O is a fixed point on BC. So during the revolution AO will always be perpendicular to BC at O.

∴ The revolving line OA will describe a plane.

Again, ∵ A is always equidistant from O,

∴ the pt. A will describe a circle of which O is the centre and OA is the radius.

Ex. 2. Prove that there cannot be more than three mutually perpendicular straight lines in space meeting at a point. [C. U. '32, '36, '48]

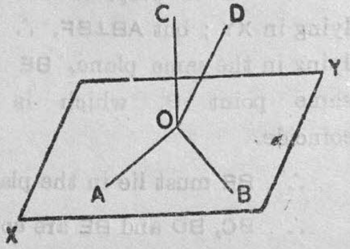


Fig. 36

Let OA, OB, OC be mutually perpendicular at the pt. O. If possible, let OD be another st. line perpendicular to OA, OB and OC at O.

Draw the plane XY passing through OA, OB.

∵ OC is perp. to OA and OB at O, ∴ OC is perpendicular to the plane XY.

Again. ∵ OD is perp. to OA and OB at O, ∴ OD is perp. to the plane XY, ∴ OC and OD are both perpendicular to the plane XY at the same point, which is impossible unless OD coincides with OC.

Hence more than three st. lines cannot be mutually perpendicular at a point.

Ex. 3. Prove that all straight lines drawn perpendicular from a given point to a system of parallel straight lines in space are coplanar.

Let AB, CD, EF··· be a system of parallel st. lines in space and let PQ, PR, PS···be respectively drawn perpendicular to them from the point P. To prove that these perpendiculars are coplanar.

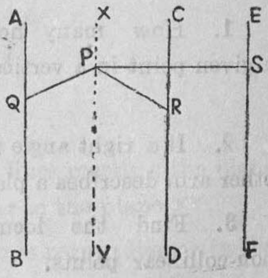


Fig. 37

Proof. Suppose XY to be drawn through P parallel to the given parallel st. lines.

∵ XY ∥ AB and PQ⊥AB, ∴ PQ⊥XY.

Similarly PR, PS,···are each perpendicular to XY at P.

Now, since PQ, PR, PS,···are perpendiculars to the same st. line XY at P, these perpendiculars are coplanar.

Ex. 4. Show that there is one and only one point equidistant from four given points which do not lie in one plane and no three of which are in one st. line. [N. U. '48]

Let A, B, C and D be four points not lying in one plane and no three of which are collinear.

It is required to prove that there can be only one point equidistant from the four points.

Proof. All points in space equidistant from A and B must lie in the plane bisecting AB at right angles. Similarly the points equidistant from B and C and those from C and D respectively lie on the two planes bisecting BC and CD at right angles.

These three planes mutually intersect in three st. lines. These three st. lines meet at only one point common to the three planes and this point is equidistant from A, B, C, D. Hence there is only one point equidistant from A, B, C, D.

Exercise 5

1. How many horizontal lines can be drawn through a given point in a vertical line and how do they lie ?

[C. U. 1916]

2. If a right angle revolves about one of its arms, then the other arm describes a plane. [H. S. '62 ; B. U. E. '64]

3. Find the locus of a point equidistant from three non-collinear points.

4. Prove that the four angular points of a horizontal rectangle are equidistant from any point on the vertical line through the centre of the rectangle.

Theorem 3

If two straight lines are parallel and if one of them is perpendicular to a plane, then the other also is perpendicular to the same plane.

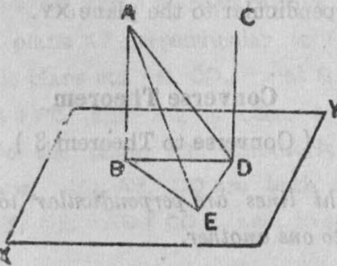


Fig. 38

Let AB and CD be two parallel st. lines meeting the plane XY at B and D and let AB be perpendicular to the plane XY.

It is required to prove that CD also is perpendicular to the plane XY.

Cons. Join BD, then BD lies in the plane XY. Through D draw $DE \perp BD$ in the plane XY. Join BE, AE, AD.

Proof. $\because$ AB is perpendicular to the plane XY,

$\therefore$ AB is perpendicular to BD and BE that lie in plane XY.

Now, $AE^2 = AB^2 + BE^2$ [$\because$ $\angle ABE = 1$ rt. angle]

$= AB^2 + BD^2 + DE^2$ [$\because$ $\angle BDE = 1$ rt. angle]

$= AD^2 + DE^2$ [$\because$ $\angle ABD = 1$ rt. angle]

$\therefore$ $\angle ADE$ is a right angle, $\therefore$ $ED \perp AD$.

But DE is perp. to BD (const.), $\therefore$ ED is perpendicular to the plane ABD containing BD and AD.

But CD, being parallel to AB, lies in the plane of BD and AD, *i.e.*,in plane ABD. $\therefore$ DE is perpendicular to the plane of AB,CD.

$\therefore$ $CD \perp ED$.

Again, $\because$ BD cuts the parallel st. lines AB and CD and $\angle$ABD is a right angle, $\therefore$ $\angle$CDB is a right angle. $\therefore$ CD$\perp$BD.

$\therefore$ CD is perpendicular to BD and DE.

But BD and DE lie in the plane XY,

$\therefore$ CD is perpendicular to the plane XY.

Converse Theorem

(Converse to Theorem 3)

If two straight lines are perpendicular to the same plane, they are parallel to one another.

As in theorem 3, it can be proved that DE is perpendicular to AD [Here write up to this from theorem 3.]

Again, $\because$ CD is perpendicular to the plane XY (by hypothesis), $\therefore$ CD$\perp$DE. $\therefore$ CD, AD, BD are coplanar.

But AB, AD, BD are coplanar, $\therefore$ AB and CD are coplanar.

Now, $\because$ $\angle ABD + \angle CDB = 1$ rt. $\angle + 1$ rt. $\angle = 2$ right angles,

$\therefore$ AB and CD are parallel.

Corollary. If AB is perpendicular to any plane XY and if from B, the foot of the perpendicular, a st. line BD is drawn perpendicular to any st. line DE in the same plane, then AD is perpendicular to DE. [Drawn Fig. of theorem 3]

Join BE, AD, AE. $\because$ AB is perpendicular to the plane XY,

$\therefore$ AB is perpendicular to BD and BE.

$\therefore$ $AE^2 = AD^2 + BE^2$ [$\because$ $\angle ABE = 1$ rt. angle]

$= AB^2 + BD^2 + DE^2$ [$\because$ $\angle BDE = 1$ rt. angle]

$= AD^2 + DE^2$ [$\because$ $\angle ABD$ is a rt. angle]

$\therefore$ $\angle$ADE is a rt. angle. $\therefore$ AD$\perp$DE.

[This theorem is known as **"The Theorem of Three Perpendiculars.**]

Miscellaneous Examples (6)

Ex. 1. Straight lines in space which are parallel to a given straight line are parallel to one another.

[C. U. '14, '19, '29, '35]

Let the st. lines AB, CD,··· be each parallel to the given st. line PQ. Draw a plane XY perpendicular to PQ through any pt. Q in it. Let this plane cut AB, CD,······at B, D,···

Proof. ∵ AB ∥ PQ and PQ is perp. to the plane XY, ∴ AB is perp. to the plane XY. Similarly CD is perp. to the plane XY. Now, ∵ AB, CD are both perpendicular to the same plane XY, ∴ AB ∥ CD ; and so on. Hence the theorem is proved.

Ex. 2. AB, AC are two st. lines intersecting at right angles, and from B a perpendicular BD is drawn to the plane of AB, AC. Show that AD is perpendicular to the line AC.

[C. U. '38]

Let *m* be the plane of AB, AC. Draw BE⊥AB, then BE ∥ AC. ∵ BD is perp. to *m*-plane (hypothesis), ∴ BD⊥BE. ∴ BE is perpendicular to both BD and AB. ∴ BE is perp. to the plane of BD, AB. ∴ AC is also perp. to that plane. Now, since AC meets AD lying on that plane at A, ∴ AC⊥AD.

Ex. 3. If perpendiculars are drawn from an external point to a system of parallel straight lines in a plane, show that their feet lie in a st. line perpendicular to the parallel lines.

[C. U. '27 ; D. U. '41]

Let AB, CD, EF,···be a system of parallel st. lines lying in the plane *m* and OP, OQ, OR, · be respectively drawn perpendiculars to them from an external point O.

It is required to prove that the feet of the perpendiculars lie in the same st. line which is perpendicular to the parallel lines AB, CD, EF···

Proof. Let XY be drawn parallel to AB, CD, EF,…… through O. ∵ AB ∥ XY and OP⊥AB, ∴ OP⊥XY. Similarly OQ, OR,⋯are perpendicular to XY at O. ∴ the perpendiculars OP, OQ, QR,⋯must lie in the same plane and that plane will be perpendicular to the plane XY. Let n be that plane. Now the plane n will intersect the given plane m in a st. line.

∵ the perpendiculars OP, OQ, OR,⋯lie in n-plane,

∴ their feet must lie on the line of intersection of m and n planes. ∴ the feet P, Q, R,……are in the same straight line.

Again, ∵ XY is parallel to AB, CD, EF,⋯and ∵ XY is perpendicular to the n-plane, ∴ each of the lines AB, CD, EF,⋯is perpendicular to the n-plane and the straight line PQR⋯is in that plane. ∴ the st. line PQR⋯, the line of collinearity, is perpendicular to the parallel lines AB, CD, EF⋯

Exercise 6

1. Draw a straight line perpendicular to a given plane from an external point.

2. From an external point P, PO is drawn perpendicular to the plane XY and LM is any straight line in the plane XY. If PQ be drawn perpendicular to LM, show that OQ is perpendicular to LM. [C. U. '43]

3. Find the locus of the foot of the perpendicular drawn from a given point upon any plane passing through a given straight line. [D. B. '24]

4. If perpendiculars are drawn from any point to a system of parallel straight lines in space, then all the perpendiculars lie in a plane perpendicular to the parallel lines. [C. U. '26]

Projection

Definitions: (1) The foot of the perpendicular drawn from a given point to a plane is called the **projection of the point** on the plane.

(2) The **projection of a line** on a plane is the locus of the foot of the perpendicular drawn from any point in the given line to the plane.

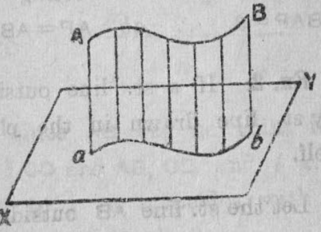


Fig. 39

The line *ab* in the figure is the **projection** of the line AB on the plane XY.

[N. B. "The projection of a st. line on a plane is a st. line". It is a theorem, but it is out of syllabus. A straight line and its projection on any plane are coplanar.]

The angle between a straight line and a plane.

The angle between a straight line and a plane is the angle between the straight line and its projection on the plane.

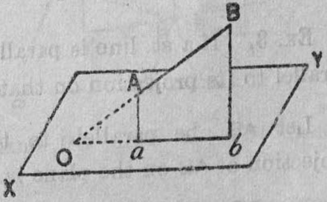


Fig. 40

Let *ab* be the projection of the straight line AB on the plane XY. Then BA and *ba* are coplanar and they must meet, produced if necessary, at some point O in the plane XY. Then the angle between AB and the plane XY is measured by the angle BO*b*.

Miscellaneous Examples (7)

Ex. 1. Find the length of the projection of of a straight line AB on a plane XY in terms of AB and the angle which AB makes with XY. [C. U. '34]

Let ab be the projection of AB on the plane XY.

It is required to find the length of ab.

Let the st. line AB make the angle θ with the plane XY.

Draw AP ∥ ab meeting Bb at P. Then AP $= ab$ and $\angle$BAP $= \theta$. ∴ $ab =$ AP $=$ AB $\cos\theta$.

Ex. 2. If a st. line outside a given plane is parallel to any st. line drawn in the plane, it is parallel to the plane itself. [C. U. '31. '33]

Let the st. line AB outside the given plane m be parallel to any st. line PQ lying in plane-m.
It is required to prove that AB is parallel to the plane m.

∵ AB ∥ PQ, ∴ AB and PQ lie in the same plane.

Let n be that plane. Then PQ must be the line of intersection of the planes m and n. ∴ AB, when produced can never meet the plane m, because if AB meets m, it must meet it in the st. line PQ, which is impossible as AB ∥ PQ.

∴ the st. line AB is parallel to the plane m.

Ex. 3. If a st. line is parallel to a plane, show that it is parallel to its projection on that plane. [C. U. '44]

Let AB be parallel to the plane m and PQ be the projection of AB on the plane m. Join AP, BQ.

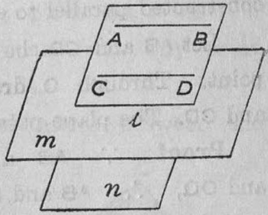
∵ PQ is the projection of AB on the plane m,

∴ each of the angles APQ and BQP is a right angle.

Again, ∵ AB is parallel to the plane m and AP, BQ are perpendiculars on the plane, ∴ $\angle$PAB and $\angle$ABQ are right angles. ∴ ABQP is a rectangle ∴ AB ∥ PQ.

Ex. 4. If a st. line AB is parallel to a plane, then any st. line CD parallel to AB is either parallel to the plane or lies in it.

[**Hints** : Let the st. line AB be parallel to the plane-m. $\because$ AB $\parallel$ CD, $\therefore$ there is only one plane (say n) through AB, CD.

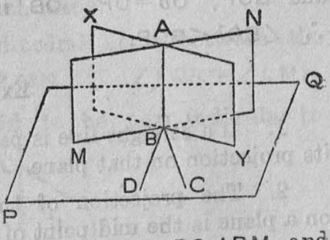


The plane n meets the plane m in the line l, or is parallel to m.

If n is parallel to m, then CD lying in n is parallel to m.

If n meets m in the line l, then since AB has no point in common with m, AB $\parallel l$. But AB $\parallel$ CD and AB, CD and l are coplanar, $\therefore$ either CD is parallel to l or coincides with l. Hence CD is either parallel to the plane m or lies in it.

Ex. 5. If two intersecting planes are each perpendicular to a third plane, their line of section is also perpendicular to that plane. [B. U. E. '64



Let the two intersecting planes MN and XY be each perpendicular to the plane PQ and let AB be the line of intersection of the planes MN and XY. To prove that AB is perpendicular to the plane PQ.

Let the planes MN and XY intersect the plane PQ in BM and BY respectively. In the plane PQ draw BC $\perp$BM and BD$\perp$BY.

$\because$ The planes MN, PQ are perpendicular to each other and BC is drawn in the plane PQ perpendicular to their line of section BM, $\therefore$ BC is perpendicular to the plane MN.

$\therefore$ BC is perpendicular to AB lying in plane MN.

Similarly, BD is perpendicular to AB.

$\therefore$ AB is perpendicular to the two intersecting lines BC and BD at B, $\therefore$ AB is perpendicular to the plane PQ containing BC and BD.

Ex. 6. Show that through a given point a plane may be constructed parallel to each of two skew lines. [C. U. '31]

Let AB and CD be the two given skew lines and O the given point. Through O draw OP and OQ parallel respectively to AB and CD. The plane passing through OP, OQ is the required plane.

Proof. $\because$ AB and CD are respectively parallel to OP and OQ, $\therefore$ AB and CD are also each parallel to the plane containing OP and OQ.

Ex. 7. The angle which a st. line makes with its projection on a plane is less than that which it makes with any other st. line which meets it in that plane. [C. U. '18, '30, '31]

Let ab be the projection of the st. line AB on the plane m and let BA and ba meet at O in the plane.

Draw $OP = Ob$ in the plane m and join BP and Pb.

Proof. $\because$ Bb is perpendicular to the m-plane, $\therefore$ $\angle BbP$ is a right angle. $\therefore$ $Bb < BP$ (hypotenuse). Now in $\triangle^s BOb$ and BOP, $Ob = OP$, OB is common to both, but $Bb < BP$, $\therefore$ $\angle BOb < BOP$.

Exercise 7

1. If a straight line is parallel to a plane, it is parallel to its projection on that plane.

2. The projection of the middle point of a straight line on a plane is the mid point of the projection. [C. U. '16]

3. Prove that the length of the projection of a straight line on a plane = the length of the straight line × the cosine of the angle which it makes with the plane.

4. Show that the projection of a straight line on a plane cannot be greater than the line. What may be the maximum length of the projection ?

5. Prove that equal and parallel straight lines have equal and parallel projections on a plane. [C. U. '23]

6. Show that if the projections of a given line on two intersecting planes be both straight lines, the given line is itself a straight line. [C. U. '26]

Dihedral Angle

The angle between two planes is the plane angle between two straight lines drawn, one in each plane, from any point in the line of intersection at right angles to it.

This plane angle is called a **Dihedral angle** between the two planes. Thus a dihedral angle is generated by two intersecting planes.

Let the two planes BC and AD intersect in the st. line AB.

From any pt. Q in AB, let QP and QR be drawn perpendicular to AB in the planes BC and AD respectively.

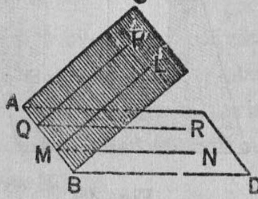


Fig. 41

Then $\angle PQR$ = the dihedral angle between the two planes.

From any other pt. M in AB st. lines ML and MN are drawn perpendicular to AB, one in each plane.

Then $\angle LMN$ measures the dihedral angle between the two planes. $\because PQ \parallel LM$ and $QR \parallel MN$, $\angle PQR = \angle LMN$.

Definition. One plane is said to be perpendicular to another when the dihedral angle between them is a right angle.

The plane BC will be perpendicular to the plane AD, if the dihedral angle PQR is a right angle.

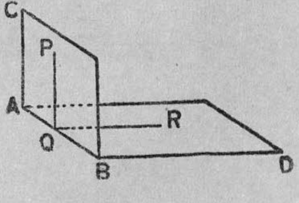


Fig. 42

[**N. B.** (i) The dihedral angle between two intersecting planes is either equal or supplementary to the angle between their normals.

(ii) If a plane intersects two parallel planes, the two corresponding dihedral angles are equal.

(iii) If a st. line is perpendicular to a plane, then any plane passing through the st. line is perpendicular to the given plane.

Let the st. line PO be perp. to the plane XY at O and let AB be any plane passing through PO and intersecting the plane XY along the st. line OB.

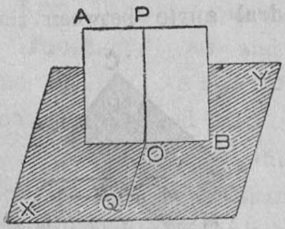


Fig 43

To prove that the plane AB is perpendicular to the plane XY.

Cons. In plane XY, draw OQ⊥OB.

Proof. ∵ PO is perp. to plane XY,

∴ PO is perp. to both OB and OQ.

∴ the st. line PO in plane AB and the st. line OQ in plane XY are each perp. to OB, the line of section of the two planes, at O. ∴ ∠POQ is the measure of the dihedral angle between the planes AB, XY. ∵ this dihedral ∠POQ is a right angle, ∴ the plane AB is perpendicular to the plane XY.]

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CHAPTER V

CO-ORDINATE GEOMETRY

1. That branch of Mathematics in which Geometry is studied by means of Algebra is known as **Co-ordinate Geometry** (Or, Analytic Geometry).

You have already learnt what is meant by the co-ordinates of a point and that the poisition of a point on a certain plane is known when its co-ordinates are known. If it is a moving point, there must exist certain relation between its co-ordinates which can be expressed as an algebraic equation.

2. (i) Co-ordinates.

You know that a plane is divided into four parts, if two straight lines at right angles to each other are drawn on it.

Let XOX′ and YOY′ be two infinite st. iines at right angles

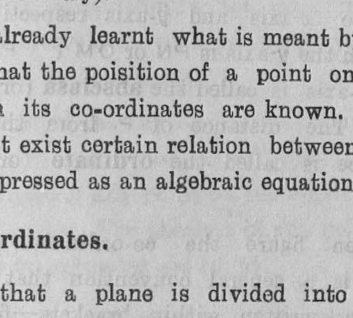


Fig. 1

to each other at the point O. As the two st. lines are fixed, the pt. O is also a fixed point.

Each of these two st. lines is called an **axis.** The st. line XOX′ is called the axis of x (or, x-axis) and the st. line YOY′ is called the axis of y (or, y-axis). The pt. O is called the origin. The distances of any point of the plane from the two axes are called its **co-ordinates.**

(ii) **Abscissa and ordinate**. The position of any point on the plane with respect to the axes can be determined. Let P be any point on the plane. From P draw PM and PN perpendiculars to x-axis and y-axis respectively. Then the distance of P from the y-axis is PN or OM ($\because$ PN$=$OM). This distance from y-axis is called the **abscissa** (or, **x co-ordinate**) of the point. The distance of P from the x-axis is PM and this distance is called the **ordinate** (or, **y co-ordinate**) of the point.

In the given figure the co-ordinates of the point P are (x, y). It is a general convention that the co-ordinates of a point are written within brackets—first the abscissa and then the ordinate with a comma between the two. Thus the point (3, 4) denotes a point whose abscissa is 3 units of length and ordinate is 4 units of length.

(iii) **Quadrant.** The two axes have divided the plane into four parts, each of which is called a **quadrant**. In the figure-(1) the portions between $\angle$XOY, $\angle$YOX$'$, $\angle$X$'$OY$'$ and $\angle$Y$'$OX are respectively called the first, the second, the third and the fourth quadrant.

(iv) **Positive and negative co-ordinates**. According to convention the distances measured in the direction of OX and OY are positive and distances measured in the opposite directions (*i.e.*, in the directions of OX$'$ and OY$'$) are negative.

Thus it is evident from the given figure that the abscissa and ordinate of any point P (x, y) in the first quadrant are both positive. In the second quadrant the abscissa of the pt. Q is negative while its ordinate is positive, *i.e.*, the co-ordinates of Q are $(-x, y)$. Both the abscissa and the ordinate of any pt. Q_1 in the third quadrant are negative, so its co-ordinates are $(-x, -y)$. The abscissa of any pt. Q_2 in the fourth quadrant is positive while its ordinate is negative. Hence its co-ordinates are $(x, -y)$.

Thus, if the co-ordinates of a point are known, the quadrant in which it lies and its position in the quadrant can easily be determined. Conversely if its position in any quadrant be known, its co-ordinates can be determined.

It is to be noted that both the abscissa and the ordinate of the origin O are zero and so the co-ordinates of the origin are $(0, 0)$. The ordinate of any point on the x-axis is zero and so its co-ordinates are $(x, 0)$. The abscissa of any point on the y-axis is zero and so its co-ordinates are $(0, y)$.

3. Cartesian co-ordinates : When the two axes XOX′ and YOY′ are at right angles to each other, the co-ordinates of any point are called the **rectangular co-ordinates.**

If the XOX′ and YOY′ are not at right angles but are inclined at any other angle, the co-ordinates with respect to these axes are called **oblique co-ordinates.** We have to study rectangular co-ordinates only.

As the system of the co-ordinates was first introduced by the philosopher Descartes, it is known as **Cartesian** co-ordinates.

4. Lengths of Segments. We shall now determine the distance between two given points or the length of any segment of a straight line.

(i) To find the distance of a point from the origin.

[Draw here the fig. 1] Let OX, OY be the rectangular axes and P be a pt. whose co-ordinates are (x, y).

We have to determine the length of OP, the distance of P from the origin O. Draw PM⊥OX and join OP.

$\because$ the co-ordinates of P are (x, y), $\therefore$ $OM = x$ and $PM = y$.

$\therefore$ $OP^2 = OM^2 + PM^2 = x^2 + y^2$, $\therefore$ $OP = \sqrt{x^2 + y^2}$.

(ii) To find the distance between two given points.

Let OX and OY be the rectangular axes. Let P and Q be the points whose co-ordinates are (x_1, y_1) and (x_2, y_2) respectively.

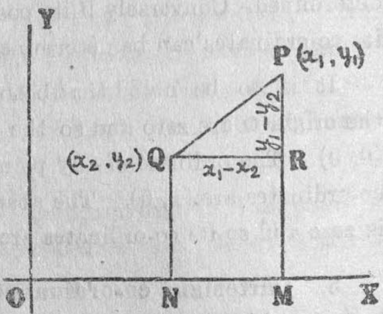


Fig. 2

To find the length of PQ [see fig. 2]. Let PM, QN be drawn perpendicular to OX. Draw QR⊥PM.

Now, $OM = x_1$, $ON = x_2$, $PM = y_1$, and $QN = y_2$.

$\therefore$ $QR = NM = OM - ON$

$= x_1 - x_2$,

and $PR = PM - RM = PM - QN = = y_1 - y_2$.

$\therefore$ In the right-angled $\triangle PQR$,

$$PQ^2 = QR^2 + PR^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2,$$

$$\therefore \quad PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

[**N. B.** (i) The above formula always holds good whether x_1, y_1, x_2, y_2 be positive or negative. Hence it can be applied to find the length PQ in whichever quadrant P or Q may lie.

(ii) The distance of P from the origin can be found by putting $x_2 = 0$, $y_2 = 0$ in the above formula, as in this case Q coincides with the origin (0, 0).]

5. Sections of a finite straight line in a given ratio.

Here we shall determine the co-ordinates of the point of intersection of a st. line where it is divided in a given ratio.

A finite st. line can be divided in two ways, either **internally** or **externally**.

(*a*) *To find the co-ordinates of a point dividing a straight line in a given ratio.*

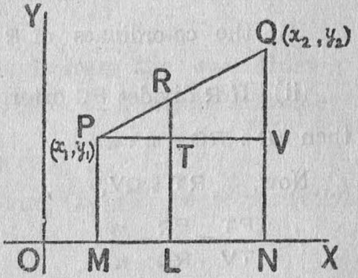


Fig. 3

Let OX, OY be the rectangular axes and P, Q be the given pts. whose co-ordinates are (x_1, y_1) and (x_2, y_2) respectively. Let PQ be divided at R in the ratio $m : n$, so that PR : RQ $= m : n$.

It is required to find the co-ordinates of R. Let the co-ordinates of R be (x, y).

(i) [In fig. 3] Suppose R divides PQ internally at R. Draw PM, QN, RL perpendicular to OX and draw PV ∥ OX, cutting RL and QN at T and V respectively.

$$\text{Now, } \because \text{ RT} \parallel \text{QV}, \quad \therefore \frac{\text{PT}}{\text{TV}} = \frac{\text{PR}}{\text{RQ}} = \frac{m}{n};$$

but $\text{PT} = \text{ML} = \text{OL} - \text{OM} = x - x_1$;

and $\text{TV} = \text{LN} = \text{ON} - \text{OL} = x_2 - x$;

$$\therefore \frac{x - x_1}{x_2 - x} = \frac{m}{n}; \quad \therefore x = \frac{mx_2 + nx_1}{m+n}.$$

Again, $\because$ △PRT and △PQV are similar,

$$\therefore \frac{\text{RT}}{\text{QV}} = \frac{\text{PR}}{\text{PQ}} = \frac{m}{m+n} \left[\because \frac{\text{RQ}}{\text{PR}} = \frac{n}{m}, \quad \therefore \frac{\text{PQ}+\text{PR}}{\text{PR}} = \frac{n+m}{m}, \right.$$

$$\left. \text{or, } \frac{\text{PQ}}{\text{PR}} = \frac{n+m}{m}, \quad \therefore \frac{\text{PR}}{\text{PQ}} = \frac{m}{m+n} \right].$$

But $\text{RT} = \text{RL} - \text{TL} = \text{RL} - \text{PM} = y - y_1$,

and $\text{QV} = \text{QN} - \text{VN} = \text{QN} - \text{PM} = (y_2 - y_1)$.

$$\therefore \quad \frac{y-y_1}{y_2-y_1}=\frac{\mathrm{RT}}{\mathrm{QV}}=\frac{m}{m+n},$$

or, $y(m+n)=m(y_2-y_1)+y_1(m+n)=my_2+ny_1$

$$\therefore \quad y=\frac{my_2+ny_1}{m+n}$$

$\therefore$ the co-ordinaes of R are $\left(\dfrac{mx_2+nx_1}{m+n},\ \dfrac{my_2+ny_1}{m+n}\right)$.

(ii) If R divides PQ externally in the ratio $m:n$ [see fig. 4], then PR : RQ $=m:n$.

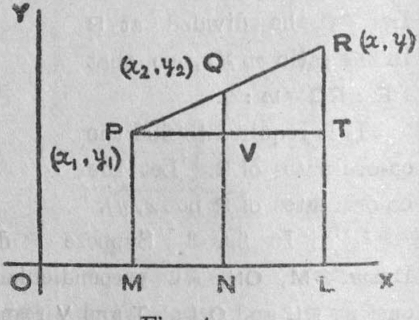


Fig. 4

Now, $\because$ RT ∥ QV,

$$\therefore \quad \frac{\mathrm{PT}}{\mathrm{TV}}=\frac{\mathrm{PR}}{\mathrm{RQ}}=\frac{m}{n};$$

But PT $=$ ML $=x-x_1$,

and VT $=$ NL $=x-x_2$,

$$\therefore \quad \frac{x-x_1}{x-x_2}=\frac{m}{n},$$

or, $x(m-n)=mx_2-nx_1$

$$\therefore \quad x=\frac{mx_2-nx_1}{m-n}.$$

Again, $\because$ $\triangle$PRT and $\triangle$PQV are similar, $\therefore$ $\dfrac{\mathrm{RT}}{\mathrm{QV}}=\dfrac{\mathrm{RP}}{\mathrm{QP}}=\dfrac{m}{m-n}$;

but RT $=y-y_1$ and QV $=y_2-y_1$, $\therefore$ $\dfrac{y-y_1}{y_2-y_1}=\dfrac{m}{m-n}$,

or, $y(m-n)=m(y_2-y_1)+y_1(m-n)=my_2-ny_1$

$$\therefore \quad y=\frac{my_2-ny_1}{m-n}.$$

$\therefore$ the co-ordinates of R are $\left(\dfrac{mx_2-nx_1}{m-n},\ \dfrac{my_2-ny_1}{m-n}\right)$.

[**N. B.** If PQ is bisected at R, i.e., if R be the middle point of PQ, then $m=n$.

Then $\dfrac{mx_2+nx_1}{m+n}=\dfrac{mx_2+mx_1}{m+m}=\dfrac{m(x_2+x_1)}{2m}=\dfrac{x_1+x_2}{2}$,

and $\dfrac{my_2+ny_1}{m+n}=\dfrac{m(y_2+y_1)}{2m}=\dfrac{y_1+y_2}{2}$. So, the co-ordinates of R, the mid pt., will be $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)\Big]$.

6. To find the area of a triangle from the co-ordinates of its vertices. (or, To find the area of a triangle formed by joining three given points).

(1) Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be respectively the co-ordinates of the vertices A, B and C of the △ABC.

To find the area of the △ABC.

Draw AM, BN and CP perpendicular to the x-axis.

Now, △ABC = trapezium AMPC + trapezium PNBC − trapezium AMNB.

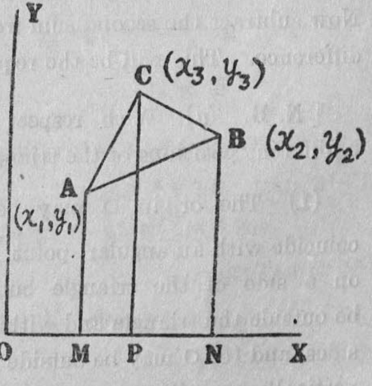


Fig. 5

∵ The area of a trapezium

$$=\tfrac{1}{2}\times \text{height}\times(\text{sum of the parallel sides}),$$

∴ The area of $\triangle ABC=\tfrac{1}{2}MP(AM+PC)+\tfrac{1}{2}PN(PC+BN)$
$-\tfrac{1}{2}(MN(AM+BN)$

$$=\tfrac{1}{2}\{(x_3-x_1)(y_1+y_3)+(x_2-x_3)(y_3+y_2) -(x_2-x_1)(y_1+y_2)\}$$

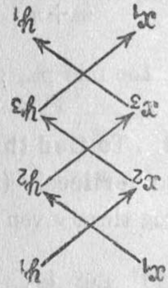
$$=\tfrac{1}{2}(x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3)$$

$$=\tfrac{1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}.$$

(2) The following artifice may be adopted in finding out the area of a triangle or any rectilineal figure.

Write the co-ordinates of the vertices, one set below the other, and repeat the first set last of all at the bottom. Then multiply downwards as indicated by the arrows (*i.e.*, x_1y_2, x_2y_3,...etc.) and take the algebraic sum of the products.

Again multiply upwards as indicated by the arrows (*i.e.*, x_2y_1, x_3y_2, etc.) and take their algebraic sum. Now subtract the second sum from the first and take half of the difference. This will be the required area.



[**N. B.** (*a*) With respect to O, the origin, there may be 5 different positions of the triangle, viz.

(1) The origin O may be within the triangle, (2) O may coincide with an angular point of the triangle, (3) O may be on a side of the triangle but not on any vertex, (4) O may be outside the triangle and within the angle included by its two sides, and (5) O may be outside the triangle and within the angle vertically opposite to any of its angles.

(*b*) According to convention, the area of the plane rectilineal figure is positive, if its angular points are taken clockwise.]

7. Condition for collinearity of three points.

This condition may be derived from the formula for the area of a triangle. If the area of the triangle having the three given points as its vertices be zero, then these points are collinear. Let the co-ordinates of the 3 points be

(x_1, y_1), (x_2, y_2) and (x_3, y_3). Then the condition for their collinearity is that

$$(x_1y_2 - x_2y_1)+(x_2y_3 - x_3y_2)+(x_3y_1 - x_1y_3)=0.$$

8. To find the area of a quadrilateral whose vertices are given.

(1) Let (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) be respectively the co-ordinates of the vertices A, B, C, D of the quadrilateral ABCD.

It is required to find the area of the quad. ABCD.

Draw AM, BN, CP, DQ perpendicular to the x-axis.

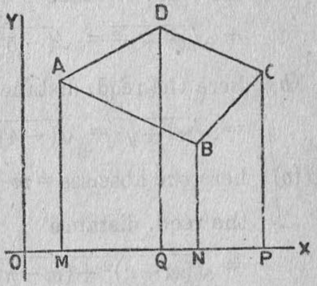


Fig. 6

Now, the quad. ABCD = trapezium AMQD + trap. DQPC − trap. AMNB − trap. BNPC

$$=\tfrac{1}{2}\mathrm{MQ}(\mathrm{AM}+\mathrm{DQ})+\tfrac{1}{2}\mathrm{QP}(\mathrm{DQ}+\mathrm{CP})-\tfrac{1}{2}\mathrm{MN}(\mathrm{AM}+\mathrm{BN})-\tfrac{1}{2}\mathrm{NP}(\mathrm{BN}+\mathrm{CP})$$

$$=\tfrac{1}{2}\{(x_4-x_1)(y_1+y_4)+(x_3-x_4)(y_4+y_3)-(x_2-x_1)(y_1+y_2)-(x_3-x_2)(y_2+y_3)\}$$

$$=\tfrac{1}{2}\{(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_4-x_4y_3)+(x_4y_1-x_1y_4)\}.$$

Cor. Similarly if the co-ordinates of the vertices of a polygon of n sides be (x_1, y_1), (x_2, y_2), $\cdots\cdots$, (x_n, y_n), then its area $=\tfrac{1}{2}\{(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+\cdots+(x_ny_1-x_1y_n)\}$.

[**N. B.** The area of a quadrilateral may be found by dividing it into two triangles by drawing any of its diagonals.]

(2) The area of a quadrilateral may also be easily found by the artifice shown in Art. 6(2).]

Examples (1)

Ex. 1. Find the distance of the following points from the origin : (*a*) $(-5, 12)$, (*b*) $(-4, -3)$, and (*c*) $\{(m+n), (m-n)\}$.

(*a*) The co-ordinates of the origin are $(0, 0)$.

$\therefore$ the reqd. distance

$$= \sqrt{x^2+y^2} = \sqrt{(-5)^2+(12)^2} = \sqrt{169} = 13.$$

(*b*) here the reqd. distance

$$= \sqrt{x^2+y^2} = \sqrt{(-4)^2+(-3)^2} = \sqrt{25} = 5.$$

(*c*) here the abscissa $= m+n$, and the ordinate $= m-n$.

$\therefore$ the reqd. distance

$$= \sqrt{(m+n)^2+(m-n)^2} = \sqrt{2m^2+2n^2} = \sqrt{2(m^2+n^2)}.$$

Ex. 2. Find the distance between the points $(0, 0)$ and $(a \cos \theta, a \sin \theta)$.

Here the pt. $(0, 0)$ is the origin, so the distance of the point $(a \cos \theta, a \sin \theta)$ from the origin is to be found.

$\therefore$ the reqd. distance

$$= \sqrt{(a \cos \theta)^2+(a \sin \theta)^2} = \sqrt{a^2(\sin^2\theta+\cos^2\theta)}$$

$$= \sqrt{a^2} \quad [\because \sin^2\theta+\cos^2\theta=1] = a.$$

Ex. 3. Find the distance between the following pairs of points :—(i) $(5, 3), (2, 2)$; (ii) $(3, -2), (-4, 3)$ and (iii) $(ax, bx), (by, -ay)$.

(i) The distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is found from the formula

$$PQ = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}.$$

$\therefore$ the reqd. distance

$$= \sqrt{(5-2)^2+(3-2)^2} = \sqrt{9+1} = \sqrt{10}.$$

(ii) Let P $(3, -2)$ and Q$(-4, 3)$ be the points [Fig. 7]
Here PR$=7$ and RQ$=5$,

$$\therefore \quad PQ=\sqrt{7^2+5^2}=\sqrt{74}.$$

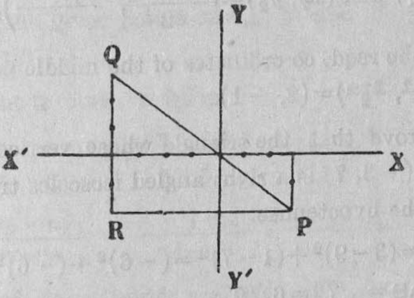


Fig. 7

Otherwise : The distance

$$=\sqrt{\{3-(-4)\}^2+\{-2-(3)\}^2}=\sqrt{(7)^2+(-5)^2}=\sqrt{74}.$$

(iii) Here the reqd. distance

$$=\sqrt{(ax-by)^2+\{bx-(-ay)\}^2}$$
$$=\sqrt{(ax-by)^2+(bx+ay)^2}$$
$$=\sqrt{(a^2+b^2)(x^2+y^2)}.$$

Ex. 4. Find the distance between the points whose co-ordinates are $(a \cos \theta, a \sin \theta)$ and $(a \cos \phi, a \sin \phi)$.

Let P and Q be the two given points.

$$\therefore \quad PQ^2=(a \cos \theta-a \cos \phi)^2+(a \sin \theta-a \sin \phi)^2$$
$$=a^2 \cos^2\theta+a^2 \cos^2\phi-2a^2 \cos \theta \cos \phi +a^2 \sin^2\theta+a^2 \sin^2\phi-2a^2 \sin \theta \sin \phi$$
$$=a^2(\cos^2\theta+\sin^2\theta)+a^2(\cos^2\phi+\sin^2\phi) -2a^2(\cos \theta \cos \phi+\sin \theta \sin \phi)$$
$$=a^2+a^2-2a^2 \cos (\theta-\phi)=2a^2-2a^2 \cos (\theta-\phi)$$
$$=2a^2\{1-\cos (\theta-\phi)\}=2a^2 \times 2 \sin^2 \tfrac{1}{2}(\theta-\phi)$$
$$=4a^2 \sin^2 \tfrac{1}{2}(\theta-\phi).$$

$\therefore$ the reqd. distance$=PQ=2a \sin\dfrac{\theta-\phi}{2}$.

Ex. 5. Find the co-ordinates of the middle point of the straight line joining the points $(6, 2)$ and $(-2, -4)$.

$\because$ the co-ordinates of the mid point of the straight line joining (x_1, y_1) and $(x_2, y_2)=\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$,

$\therefore$ here the reqd. co-ordinates of the middle point $=\left(\frac{6-2}{2}, \frac{2-4}{2}\right)=(2, -1)$.

Ex. 6. Prove that the triangle whose vertices are A $(3, 1)$, B $(9, 7)$ and C$(-3, 7)$ is a right-angled isosceles triangle and find the length of the hypotenuse.

Here $AB^2=(3-9)^2+(1-7)^2=(-6)^2+(-6)^2=72$,

$\therefore AB=\sqrt{72}=6\sqrt{2}$.

Again, $AC^2=\{3-(-3)\}^2+(1-7)^2=72$,

$\therefore AC=\sqrt{72}=6\sqrt{2}$.

$\therefore AB=AC$ and hence the triangle is isosceles.

Now, $BC^2=\{9-(-3)\}^2+(7-7)^2=(12)^2=144$.

$\because AB^2=72$ and $AC^2=72$, $\therefore AB^2+AC^2=144=BC^2$,

$\therefore \angle A$ is a rt. angle. $\therefore$ it is a right-angled triangle.

Hence $\triangle ABC$ is a right-angled isosceles triangle and its hypotenuse $BC=12$.

Ex. 7. Find the co-ordinates of the point which divides the straight line joining the points $(8, 12)$ and $(-2, 7)$ internally in the ratio $3 : 2$.

Let (x, y) be the required co-ordinates.

Now from the formula

$$x=\frac{mx_2+nx_1}{m+n}, \text{ and } y=\frac{my_2+ny_1}{m+n},$$

we have $x=\frac{3\times-2+2\times8}{3+2}=\frac{-6+16}{5}=2$,

and $y=\frac{3\times7+2\times12}{3+2}=\frac{21+24}{5}=9$.

$\therefore$ the required co-ordinates are $(2, 9)$.

Ex. 8. Find the co-ordinates of a point which divides the straight line joining the points (4, 5) and (7, −1) externally in the ratio 4 : 3.

Let the co-ordinates of the point be (x, y); here the co-ordinates of the given points are (4, 5) and (7, −1) and the given ratio = 4 : 3.

∴ from the formula we have

$$x=\frac{mx_2-nx_1}{m-n}=\frac{4\times 7-3\times 4}{4-3}=\frac{16}{1}=16,$$

$$\text{and } y=\frac{my_2-ny_1}{m-n}=\frac{4\times -1-3\times 5}{4-3}=-\frac{19}{1}=-19.$$

∴ the reqd. co-ordinates are (16, −19).

Ex. 9. Show that the st. line joining the points (4, 3) and (8, 6) passes through the origin.

The origin is the point (0, 0). The st. line joining the points (4, 3) and (8, 6) will pass through the origin, if the points (4, 3), (8, 6) and (0, 0) are collinear.

The condition for this collinearity is

$$(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)=0.$$

Here the pts. are (0, 0), (4, 3) and (8, 6),

$$\therefore \text{ here } (x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)$$
$$=(0\times 8-0\times 4)+(4\times 6-8\times 3)+(8\times 0-0\times 6)$$
$$=0+(24-24)+0=0.$$

∴ the three points are in the same st. line, *i.e.*, the straight line joining the points (4, 3) and (8, 6) passes through the origin.

[**Alt. Proof.**] Let the co-ordinates of the pt. P and Q be (4, 3) and (8, 6). The origin is the pt. (0, 0).

$$\text{Here, } OP=\sqrt{4^2+3^2}=\sqrt{25}=5,$$
$$OQ=\sqrt{8^2+6^2}=\sqrt{100}=10,$$

and $PQ=\sqrt{(4-8)^2+(3-6)^2}=\sqrt{16+9}=\sqrt{25}=5.$

$\therefore \quad OP+PQ=10=OQ.$

Hence, O, P and Q are in the same straight line.

[**N. B.** If $OP+PQ=OQ$, then O, P, Q must be in the same straight line. If not, then OPQ must be a triangle and in this triangle the sum of the sides OP, PQ is equal to the third side OQ, which is absurd.]

Ex. 10. Find the ratio in which the point (5, 4) divides the join of (3, 2) and (6, 5).

Let the required ratio be $m : n$.

$\because$ the point has divided the straight line joining (3, 2) and (6, 5) in the ratio $m : n$,

$\therefore$ from the formula $x=\dfrac{mx_2+nx_1}{m+n}$ we have

$$5=\frac{m\times 6+n\times 3}{m+n}, \quad \text{or,} \quad 6m+3n=5m+5n,$$

or, $m=2n$, $\therefore \quad \dfrac{m}{n}=\dfrac{2}{1}$. $\therefore$ the reqd. ratio $=2:1$.

[**N. B.** (i) Here if $\dfrac{m}{n}$ be negative, *i.e.*, if $\dfrac{m}{n}=-\frac{2}{1}$, then we shall understand that the join is divided externally in the ratio 2 : 1.

(ii) Here the same ratio is obtained from $y=\dfrac{my_2+ny_1}{m+n}$.]

Ex. 11. If the point (x, y) be equidistant from the points (2, 3) and (− 1, 2), then will $3x+y=4$.

The distance between the points (x, y) and (2, 3)

$$=\sqrt{(x-2)^2+(y-3)^2},$$

and the distance between the points (x, y) and (− 1, 2)

$$=\sqrt{(x+1)^2+(y-2)^2}.$$

$\because$ These two distances are equal (hyp.),

$\therefore$ $\sqrt{(x+1)^2+(y-2)^2}=\sqrt{(x-2)^2+(y-3)^2}$

or, $(x+1)^2+(y-2)^2=(x-2)^2+(y-3)^2$

or, $x^2+2x+y^2-4y+5=x^2-4x+y^2-6y+13$,

or, $6x+2y=8$, $3x+y=4$.

Ex. 12. The square of the distance between the points (3, 5) and $(x, 4)$ is 17 ; find the abscissa of the unknown point.

The distance between the pts. (3, 5) and $(x, 4)$

$= \sqrt{(3-x)^2+(5-4)^2}=\sqrt{(3-x)^2+1}$,

$\therefore$ the sq. of the distance $=(3-x)^2+1$.

$\therefore$ $(3-x)^2+1=17$, or; $(3-x)^2=16$, or, $3-x=\pm 4$.

$\therefore$ $x=3\mp 4=-1$ or 7.

$\therefore$ the required abscissa $=-1$ or 7.

Ex. 13. Prove that the points (3, 3), $(-3, -3)$ and $(-3\sqrt{3}, 3\sqrt{3})$ are the vertices of an equilateral triangle.

Let, A, B and C respectively be the given points.

Now, $AB^2=(3+3)^2+(3+3)^2=72$, $\therefore$ $AB=6\sqrt{2}$;

$BC^2=(-3+3\sqrt{3})^2+(-3-3\sqrt{3})^2$

$=9+27-18\sqrt{3}+9+27+18\sqrt{3}=72$,

$\therefore$ $BC=6\sqrt{2}$;

and $CA^2=(-3\sqrt{3}-3)^2+(3\sqrt{3}-3)^2=72$, $\therefore$ $CA=6\sqrt{2}$.

$\therefore$ $AB=BC=CA$. Hence $\triangle ABC$ is equilateral and the three given points are its three vertices.

Ex. 14. Find the area of the triangle whose vertices are (a, bc), (b, ca) and (c, ab).

$\because$ $\triangle$ (area) $=\frac{1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}$,

$\therefore$ the reqd. area $=\frac{1}{2}\{a(ca-ab)+b(ab-bc)+c(bc-ca)\}$

$=\frac{1}{2}\{a^2(c-b)+b^2(a-c)+c^2(b-a)\}$

$=\frac{1}{2}(a-b)(b-c)(c-a)$.

Ex. 15. Find the area of the triangle whose vertices are $(\cos\theta, \sin\theta)$, $(\cos 3\theta, \sin 3\theta)$ and $(0, 0)$.

$\because$ the area of a triangle

$$=\tfrac{1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}$$

$\therefore$ the reqd. area

$$=\tfrac{1}{2}\{\cos\theta(\sin 3\theta-0)+\cos 3\theta(0-\sin\theta)+0(\sin\theta-\sin 3\theta)\}$$

$$=\tfrac{1}{2}\{\sin 3\theta\cos\theta-\cos 3\theta\sin\theta+0\}$$

$$=\tfrac{1}{2}(\sin 3\theta\cos\theta-\cos 3\theta\sin\theta)$$

$$=\tfrac{1}{2}\sin(3\theta-\theta)=\tfrac{1}{2}\sin 2\theta.$$

Ex. 16. Find the area of the quadrilateral the co-ordinates of whose angular points, taken in order, are $(1, 2)$, $(3, 4)$, $(5, -1)$ and $(4, -3)$.

Here $x_1=1$, $x_2=3$, $x_3=5$, $x_4=4$ and

$y_1=2$, $y_2=4$, $y_3=-1$, $y_4=-3$.

$$\therefore\ \Delta=\tfrac{1}{2}\{1\times4-3\times2)+(3\times-1-5\times4)+(5\times-3-4\times-1)+(4\times2-1\times-3)\}$$

$$=\tfrac{1}{2}(-2-23-11+11)=-12\tfrac{1}{2}.$$

$\therefore$ the reqd. area $=12\tfrac{1}{2}$ units of area.

✓**Ex. 17.** Show that the three points $(a, 0)$, $(0, b)$ and $(1, 1)$ are collinear, if $a+b=ab$.

The given points will be collinear, if the area of the triangle having the given points as its vertices be zero.

Now, the area of the triangle

$$=\tfrac{1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}$$

$$=\tfrac{1}{2}\{a(b-1)+0(1-0)+1(0-b)\}=\tfrac{1}{2}(ab-a-b).$$

Hence, the three given points will be collinear,

if $\tfrac{1}{2}(ab-a-b)=0$, *i.e.*, if $ab-a-b=0$,

i.e., if $a+b=ab$.

Ex. 18. (*a*) Show that the four points $(5, 2)$, $(3, 7)$, $(-1, 4)$ and $(1, -1)$ are the angular points of a parallelogram.

Let the co-ordinates of the pts. A, B, C, D be $(5, 2)$, $(3, 7)$, $(-1, 4)$ and $(1, -1)$ respectively.

If the co-ordinates of the middle pt. of AC be (h, k),

then, $h=\frac{5-1}{2}=2$, and $k=\frac{2+4}{2}=3$.

Again, if the co-ordinates of the middle pt. of BD be (h_1, k_1), then $h_1=\frac{3+1}{2}=2$, and $k_1=\frac{7-1}{2}=3$.

$\therefore$ The diagonals AC and BD have the same middle pt., *i.e.*, the diagonals AC and BD of the quadrilateral, having the given pts. as its angular points, bisect each other.

Hence, the quadrilateral is a parallelogram.

[**N. B.** From geometry we know that a quadrilateral is a parallelogram if its diagonals bisect each other.]

(*b*) Show that the four points A $(3, 3)$, B $(5, 5)$, C $(6, 4)$ and D $(4, 2)$, when joined in order, make up a rectangle.

Here, $AB^2=(3-5)^2+(3-5)^2=8$,

and $CD^2=(6-4)^2+(4-2)^2=8$, $\therefore$ $AB=CD$.

Again, $BC^2=(5-6)^2+(5-4)^2=2$,

and $DA^2=(4-3)^2+(2-3)^2=2$, $\therefore$ $BC=DA$.

Now, $AC^2=(3-6)^2+(3-4)^2=10$,

and $BD^2=(5-4)^2+(5-2)^2=10$, $\therefore$ $AC=BD$.

We find here that the opposite sides of the quadrilateral are equal and its diagonals also are equal. Hence the quadrilateral is a rectangle.

[**N. B.** To prove a quadrilateral to be a square, you should show that its sides are equal and its diagonals also are equal. To prove a quadrilateral to be a rhombus, show that its sides are equal but its diagonals are unequal.]

Ex. 19. Prove that the join of the middle points of two sides of a triangle is equal to half the third side.

Suppose the co-ordinates of the vertices A, B, C of the triangle to be (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively and let P, Q be the mid points of AB and AC respectively.•

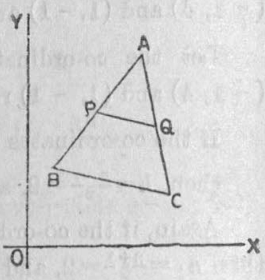


Fig. 8

To prove that $PQ = \frac{1}{2}BC$.

The co-ordinates of P are

$$\left(\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2}\right);$$

and those of Q are $\left(\frac{x_1+x_3}{2},\ \frac{y_1+y_3}{2}\right)$.

$$\therefore\quad PQ = \sqrt{\left(\frac{x_1+x_2}{2} - \frac{x_1+x_3}{2}\right)^2 + \left(\frac{y_1+y_2}{2} - \frac{y_1+y_3}{2}\right)^2}$$

$$= \sqrt{\tfrac{1}{4}(x_2-x_3)^2 + \tfrac{1}{4}(y_2-y_3)^2}$$

$$= \tfrac{1}{2}\sqrt{(x_2-x_3)^2 + (y_2-y_3)^2}.$$

Again, $BC = \sqrt{(x_2-x_3)^2 + (y_2-y_3)^2}$. $\therefore\quad PQ = \frac{1}{2}BC$.

Ex. 20. Find the co-ordinates of the centroid of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3).

Let the co-ordinates of the vertices A, B and C of the $\triangle ABC$ be (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively. [Fig. 9]

Let D be the middle point of BC and join AD. Let AD be divided at G in the ratio 2 : 1, then G is the centroid of the triangle. To find the co-ordinates of G.

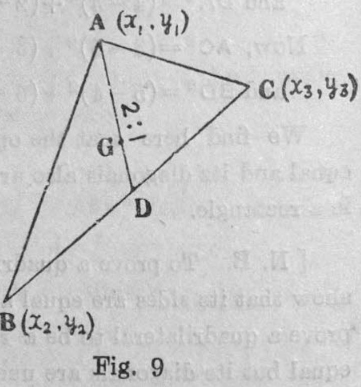


Fig. 9

Here, the co-ordinates of the mid point D are

$$\{\tfrac{1}{2}(x_2+x_3),\ \tfrac{1}{2}(y_2+y_3)\}.$$

Let (x, y) be the co-ordinates of G.

$\because$ G divides the straight line joining $A(x_1, y_1)$ and $D\left(\dfrac{x_2+x_3}{2}, \dfrac{y_2+y_3}{2}\right)$ in the ratio 2 : 1,

$$\therefore \quad x=\frac{2\times\frac{1}{2}(x_2+x_3)+1\times x_1}{2+1}=\tfrac{1}{3}(x_1+x_2+x_3),$$

$$\text{and}\quad y=\frac{2\times\frac{1}{2}(y_2+y_3)+1\times y_1}{2+1}=\tfrac{1}{3}(y_1+y_2+y_3).$$

$\therefore$ The required co-ordinates =

$$\{\tfrac{1}{3}(x_1+x_2+x_3),\ \tfrac{1}{3}(y_1+y_2+y_3)\}.$$

Ex. 21. Prove analytically that, in a triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median that bisects the third side. (Apollonius' Theorem)

Let AD be a median of the triangle ABC. To prove that

$AB^2+AC^2=2(AD^2+BD^2)$.

Let $BC=2l$. Now, if BC be taken as the x-axis and its mid pt. D be taken as the origin, then the co-ordinates of B and C are respectively $(-l, 0)$ and $(l, 0)$. Suppose the co-ordinates of A are (α, β).

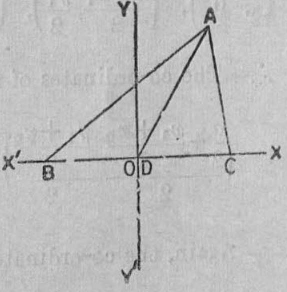


Fig. 10

$$\therefore \quad AB^2=(\alpha+l)^2+\beta^2,\ AC^2=(\alpha-l)^2+\beta^2,\ AD^2=\alpha^2+\beta^2$$

and $BD^2=(-l)^2=l^2$.

$$\therefore \quad AB^2+AC^2=(\alpha+l)^2+\beta^2+(\alpha-l)^2+\beta^2$$

$$=2(\alpha^2+l^2)+2\beta^2=2(\alpha^2+\beta^2)+2l^2=2(AD^2+BD^2).$$

Ex. 22. Prove that the lines joining the middle points of the opposite sides of a quadrilateral bisect each other.

[C. U. 1958]

Let P, Q, R, S be respectively the middle points of the sides AB, BC, CD, DA of the quadrilateral ABCD.

To prove that PR and QS bisect each other.

Take AB as the x-axis and A as the origin and suppose $AB = a$.

Then the co-ordinates of A and B are $(0, 0)$ and $(a, 0)$ respectively. Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of C and D respectively.

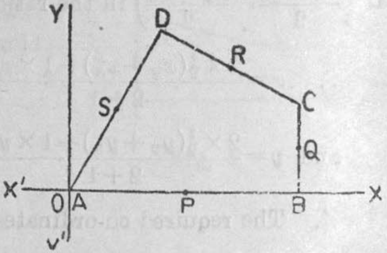


Fig. 11

Then the co-ordinates of P, Q, R and S are respectively

$$\left(\frac{a}{2}, 0\right), \left(\frac{a+x_1}{2}, \frac{y_1}{2}\right), \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \text{ and } \left(\frac{x_2}{2}, \frac{y_2}{2}\right).$$

$\therefore$ The co-ordinates of the middle pt. of PR are

$$\left(\frac{\frac{a}{2}+\frac{x_1+x_2}{2}}{2}, \frac{\frac{y_1+y_2}{2}}{2}\right) \text{ or } \left(\frac{a+x_1+x_2}{4}, \frac{y_1+y_2}{4}\right).$$

Again, the co-ordinates of the middle pt. of QS are

$$\left(\frac{\frac{a+x_1}{2}+\frac{x_2}{2}}{2}, \frac{\frac{y_1}{2}+\frac{y_2}{2}}{2}\right) \text{ or } \left(\frac{a+x_1+x_2}{4}, \frac{y_1+y_2}{4}\right).$$

$\because$ The co-ordinates of the middle pts. of PR and QS are the same, $\therefore$ they bisect each other at the same point. Hence the problem is proved.

Exercise 1

1. Find the distance of the following points from the origin :—

(i) $(12, 5)$ (ii) $(-3, 4)$ (iii) $(-8, 12)$

(iv) $(-6, 8)$ (v) $\{(a+b), (a-b)\}$

2. Find the distance between the following pairs of points :—

(*a*) $(5, 2), (2, 3)$ (*b*) $(26, 10), (2, 3)$

(*c*) $(8, 12), (-4, 7)$ (*d*) $(-3, -4), (5, -10)$

(*e*) $(m, 0), (0, n)$ (*f*) $(a+b, c-d), (a-b, c+d)$

(*g*) $(\cos\theta, \sin\theta), (\sin\theta, \cos\theta)$ (*h*) $(a, -b), (-a, b)$

3. Find the co-ordinates of the mid points of the st. lines joining the following pairs of points :—

(i) $(5, 0)$ and $(0, 7)$ (ii) $(-2, -4)$ and $(6, 2)$

(iii) $(4, -2)$ and $(3, -5)$.

4. Find the co-ordinates of the points which divide the st. lines joining the following pairs of points in the given ratio :—

(*a*) $(6, -10)$ and $(-4, 14)$, ratio 3 : 4 (internally)

(*b*) $(3, 5)$ and $(-2, -7)$, ratio 3 : 2 (internally)

(*c*) $(-1, 2)$ and $(4, -5)$, ratio 2 : 3 (externally)

(*d*) $(3, 2)$ and $(6, 5)$, ratio 2 : 1 (externally)

5. Find the co-ordinates of the point P which divides the st. line joining A $(1, 2)$ and B $(4, 3)$ so that $AP = 2PB$.

6. Find the ratio in which

(i) the point $(-2, 2)$ divides the st. line joining the points $(-4, 6)$ and $(\frac{1}{2}, -3)$;

(ii) the point $(1, 3)$ divides the join of $(4, 6)$ and $(3, 5)$.

7. If the distance between the points $(x, 7)$ and $(2, 3)$ be 5, find the value of x.

8. If the distance between the points (11, 3) and (3, y) be 10, find the ordinate of the second point.

9. The square of the distance between the points (12, 5) and (x, 3) is 29 ; find the abscissa of the unknown point.

10. Prove that the points (2, 2), (-2, -2) and ($-2\sqrt{3}$, $2\sqrt{3}$) are the vertices of an equilateral triangle.

11. Prove that the triangle formed by joining the points (1, 4), (8, 8) and (4, 1) is isosceles.

12. Prove that the points ($2a$, $4a$), ($2a$, $6a$) and ($2a+\sqrt{3}a$, $5a$) are the vertices of an equilateral triangle whose side is $2a$. [C. U. (B. Sc.) '52]

13. Show that (0, 0), (2, 1), (-1, 7) and (-3, 6) are the vertices of a rectangle.

14. Prove that the points (3, 4), (-1, 7) and (-3, -4) are the vertices of a right-angled triangle.

15. Show that the st. line joining the points (-4, -3) and (8, 6) passes through the origin.

16. Verify that the points (1, 5), (3, 14) and (-1, -4) are collinear. [C. U.]

17. Show that the three points ($3a$, 0), (0, $3b$) and (a, $2b$) are collinear. [C. U. (B. Sc.) '24]

18. If (x, y) is equidistant from (4, 7) and (-5, 8), show that $9x-y+12=0$.

19. If (x, y) is equidistant from (2, 4) and (-3, 3), show that $5x+y=1$.

20. If the point (x, y) is equidistant from (5, 0), (0, 5) and (3, 4), show that $x=0$, $y=0$.

21. Find the co-ordinates of the point equidistant from (-3, 3), (2, 1) and (5, 3).

22. Find the co-ordinates of the point equidistant from (5, 4), (3, 6) and (1, 4).

23. Find the condition that (x, y) should be equidistant from (2, 3) and (- 1, 2).

24. Find the area of the triangle whose vertices are :—

(i) $(2, -2)$, $(4, 2)$ and $(-1, 3)$

(ii) $(3, 5)$, $(-2, 4)$ and $(5, 2)$

(iii) $(8, 9)$, $(2, 6)$ and $(9, 2)$

(iv) $(1, 2)$, $(3, 0)$ and the origin

(v) $(\cos\theta, \sin\theta)$, $(\cos 2\theta, \sin 2\theta)$, $(0, 0)$

(vi) $(a, b+c)$, $(b, c+a)$, $(c, a+b)$ [C. U. 1958]

25. Find the distance between $(-2, 3)$ and $(3, -1)$ and the co-ordinates of the point of trisection that is nearer to $(-2, 3)$. [J. B. A.]

26. Find the centroid of the triangle whose vertices are $(1, 7)$, $(-4, 3)$ and $(6, -1)$.

27. Find the centroid of the triangle whose vertices are $(3, -4)$, $(4, 7)$ and $(2, 9)$.

28. Find the lengths of the medians of the triangle whose vertices are (2, 0), (4, 4) and (6, 2).

29. The area of the triangle formed by joining the points $(5, -1)$, $(x, 6)$ and $(1, 3)$ is 10 square units. Find x.

30. If the points $(0, -4)$, $(-1, y)$ and $(3, 2)$ are in the same straight line, find y.

31. In △ABC, AD bisects BC at D and is divided at G in the ratio 2 : 1. Prove that the straight lines drawn from B and C through G to the opposiae sides are divided in the same ratio.

32. Find the area of the quadrilaterals the co-ordinates of whose angular points, taken in order, are

(i) $(2, -1), (-1, 3), (3, -3), (5, 2)$

(ii) $(1, 2), (-2, 1), (2, -1), (4, 1)$

(iii) $(0, 0), (3, 1), (4, 2), (1, 5)$

(iv) $(7, 2), (5, 5), (4, 9), (1, 3)$.

33. Find the centre and radius of the circumscribed triangle whose vertices are $(8, 4)$, $(7, 7)$ and $(3, 9)$.

34. Show that the origin is the centroid of the triangle whose vertices are $(a-b, b-c)$, $(-a, -b)$ and (b, c).

35. (*a*) Prove that the points $(7, 3)$, $(9, 6)$, $(10, 12)$ and $(8, 9)$ when joined, taken in order, will form a parallelogram.

(*b*) Show that the four points $(1, 2)$, $(4, 6)$, $(-4, 12)$ and $(-7, 8)$, when joined in order, form a rectangle.

(*c*) Verify that the figure, formed by joining the four points A $(2, 5)$, B $(5, 9)$, C $(9, 12)$ and D $(6, 8)$, is a rhombus.

(*d*) Justify that the four points P$(-2, -7)$, Q$(2, -4)$, R$(-1, 0)$ and S$(-5, -3)$ are the vertices of a square.

36. If the figure formed by joining the four points (α_1, β_1), (α_2, β_2), (α_3, β_3) and (α_4, β_4), taken in order, be a parallelogram, then prove that

$$\alpha_1+\alpha_3=\alpha_2+\alpha_4 \text{ and } \beta_1+\beta_3=\beta_2+\beta_4.$$

37. The line joining A $(b \cos \alpha, b \sin \alpha)$ and B $(a \cos \beta, a \sin \beta)$ is produced to the point M (x, y) so that AM and BM are in the ratio of b to a; prove that

$$x+y \tan \frac{\alpha+\beta}{2}=0.$$ [C. U. 1959 compl.]

38. The co-ordinates of A, B, C are $(6, 3)$, $(-3, 5)$ and $(4, -2)$ respectively and P is the point (x, y); show that

$$\frac{\triangle PBC}{\triangle ABC}=\frac{x+y-2}{7}.$$ [C. U.]

39. The co-ordinates of A, B, C, D are respectively (6, 3), (– 3, 5), (4, – 2) and $(x, 3x)$ and

$$\frac{\text{area of } \triangle DBC}{\text{area of } \triangle ABC}=\tfrac{1}{2}\ ;\ \text{find } x.$$ [C. U. 1949]

40. Prove the following analytically :—

(*a*) If G be the centroid of a triangle ABC, then

(i) $3(GA^2+GB^2+GC^2)=BC^2+CA^2+AB^2$

(ii) $\triangle GBC=\frac{1}{3}\triangle ABC.$

(*b*) If D, E, F be the middle points of the sides BC, CA, AB respectively of a triangle ABC, then prove that

$$\triangle ABC=4\triangle DEF.$$

(*c*) Prove that the lines joining the middle points of the adjacent sides of a quadrilateral form a parallelogram.

LOCUS

9. Locus and its equation.

You know that if a point moves in accordance with some given condition or conditions, the path traced out by it is called its **locus.**

The given condition which the point has always to satisfy may be expressed by **co-ordinates.**

The Equation of the locus: The equation of the locus of a moving point is the equation that is satisfied by the co-ordinates of every point on the locus and by no other point.

You know from Geometry that (i) the locus of a point which moves so as to be always equidistant from two fixed points is the perpendicular bisector of the straight line joining the fixed points ;

(ii) if the point moves so as to be always equidistant from two intersecting straight lines, its locus is the pair of straight lines bisecting the angles between them ;

(iii) if the point moves so as to be always equidistant from a fixed point, its locus is the circumference of a circle ; etc.

9. (*a*) To express the condition by co-ordinates,

or, to **find the equation of the locus :**

The geometrical conditions may be expressed by means of co-ordinates.

Suppose the geometrical condition of the movement of a variable point is that its distance from the *y-axis* is always twice its distance from the *x-axis*.

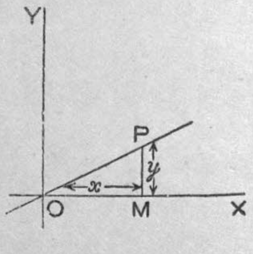


Fig. 12

It is required to find the equation of the locus of the point.

Let OX, OY be respectively the x- and y-axis [Fig. 12]. Let one position of the moving point be P and its co-ordinates be (x, y). So, here we have $x = 2y$.

Since in any other position also of the pt. P, its abscissa (x) is twice its ordinate (y), we have $x = 2y$.

$\therefore$ The equation of the locus of the point is $x = 2y$.

Hence, to find the equation of a locus (i) first draw a figure representing the given condition, (ii) in it take the co-ordinates (x, y) of one position of the moving point, and then (iii) express the given condition in terms of x, y which in its simplest form is the required equation.

[**N. B.** The equation thus obtained will be satisfied by the co-ordinates of any position of the moving point, and not by the co-ordinates of any point outside the locus.]

Examples (2)

Ex. 1. Find the equation of the locus of a point which is always equidistant from the points (3, 4) and (5, 6).

Let A and B be the points (3, 4) and (5, 6) and let P be a position of the moving point and let (x, y) be the co-ordinates of P.

Now, $AP^2 = (x-3)^2 + (y-4)^2$, and $BP^2 = (x-5)^2 + (y-6)^2$.

$\because$ from the given condition $AP = BP$,

$\therefore \quad (x-3)^2 + (y-4)^2 = (x-5)^2 + (y-6)^2$,

or, $x + y = 9$ [simplifying] which is the reqd. equation.

Ex. 2. Find the equation of the locus of a point if its distance from the x-axis is double its distance from the point $(1, 1)$. [C. U. (B. Sc.) '38]

Let (x, y) be the co-ordinates of any position of the moving point P and the given point be A $(1, 1)$.

Now, the distance of P from the x-axis $= y$, and the distance of P (x, y) from A $(1, 1) = \sqrt{(x-1)^2+(y-1)^2}$.

$\therefore$ From the given condition $y = 2\sqrt{(x-1)^2+(y-1)^2}$,

or, $y^2 = 4\{(x-1)^2+(y-1)^2\}$,

or, $y^2 = 4x^2 - 8x + 4y^2 - 8y + 8$,

$\therefore$ $4x^2 + 3y^2 - 8x - 8y + 8 = 0$, which is the reqd. equation.

Ex. 3. The point P and Q are $(-4, 0)$ and $(-1, 0)$ respectively. A point A moves in such a way that AP : AQ $= 2 : 1$. Find the locus of A.

Let (x, y) be the co-ordinates of the moving pt. A.

Then, AP $= \sqrt{(x+4)^2+y^2}$ and AQ $= \sqrt{(x+1)^2+y^2}$

$\therefore$ from the given condition we have $\dfrac{\sqrt{(x+4)^2+y^2}}{\sqrt{(x+1)^2+y^2}} = \dfrac{2}{1}$,

or, $\dfrac{(x+4)^2+y^2}{(x+1)^2+y^2} = \dfrac{4}{1}$, or, $4(x+1)^2 + 4y^2 = (x+4)^2 + y^2$,

$\therefore$ (Simplifying) $x^2 + y^2 = 4$, which is the reqd. equation.

Ex. 4. The co-ordinates of two fixed points A and B are respectively $(-2, 4)$ and $(6, 8)$. A point P moves so that the area of the triangle PAB is always 10. Find the equation to the locus of P.

Let (x, y) be the co-ordinates of the moving pt. P.

Now, the area of $\triangle$PAB $= \frac{1}{2}\{x(4-8) + (-2)(8-y) + 6(y-4)\}$

$= \frac{1}{2}\{8y - 4x - 40\} = 4y - 2x - 20.$

$\therefore$ from the given condition, $4y - 2x - 20 = 10$,

$\therefore$ $x - 2y + 15 = 0$, which is the equation required.

Ex. 5. A straight line moves such that the sum of the reciprocals of its intercepts on the axes is constant. Prove that the line passes through a fixed point. [Bombay, 1935]

Let the equation of the moving st. line be $\frac{x}{a}+\frac{y}{b}=1$.

Let its intercepts on the x-axis and y-axis be a and b respectively.

Now, from the given condition, $\frac{1}{a}+\frac{1}{b}$ is constant.

Suppose, $\frac{1}{a}+\frac{1}{b}=\frac{1}{k}\Big($ where k is a constant $\Big)$ or, $\frac{k}{a}+\frac{k}{b}=1$.

Comparing it with the above equation, we find that the st. line $\left(\frac{x}{a}+\frac{y}{b}=1\right)$ passes through the pt. (k, k), which is a fixed pt. (k being a constant).

Hence, the st. line passes through a fixed point.

Ex. 6. A straight line moves so that the sum of the intercepts made by it on the axes is always constant. Find the locus of the middle point of the intercept between the axes.

Suppose, in one position of the moving straight line, (α, β) are the co-ordinates of the middle point of its intercept between the axes. Then, evidently, the intercept made by it on the *x-axis* is 2α and that on the *y-axis* is 2β.

$\therefore$ from the given condition we have

$$2\alpha+2\beta=\text{constant}=2k \text{ (suppose)}, \quad \therefore \quad \alpha+\beta=k.$$

Hence, evidently the equation of the locus of the moving point is $x+y=k$.

Ex. 7. A and B being fixed points $(a, 0)$ and $(-a, 0)$ respectively, find the locus of P when $PB^2+PC^2=2PA^2$, C being the point $(d, 0)$. [Utkal. 1948]

Let the co-ordinates of P be (x, y).

Now, $PB^2=(x+a)^2+y^2$, $PC^2=(x-d)^2+y^2$

and $PA^2=(x-a)^2+y^2$.

$\therefore$ from the given condition, $(x+a)^2+y^2+(x-d)^2+y^2$

$$=2(x-a)^2+2y^2,$$

or, $2ax+a^2-2dx+d^2=-4ax+2a^2$,

or, $2(3a-d)x+d^2-a^2=0$, which is the reqd. equation.

Ex. 8. Given the base and the difference of the squares of the sides of a triangle. Find the equation to the locus of the vertex. [C. U. 1944]

Let the given base be $BC=2a$. Taking the base as the x-axis and its middle point O as the origin, we have $(-a, 0)$ and $(a, 0)$ as the co-ordinates of B and C respectively.

Let (x, y) be the co-ordinates of the moving vertex A.

$\therefore$ $AB^2=(x+a)^2+y^2$ and $AC^2=(x-a)^2+y^2$.

So, by the given condition, $AB^2-AC^2=$ constant.

$\therefore$ $\{(x+a)^2+y^2\}-\{(x-a)^2+y^2\}=\text{constant}=k$ (suppose)

or, $4ax=k$. $\therefore$ $x=\frac{k}{4a}$, this is the required equation.

Exercise 2

1. Find the equation to the locus of a point which moves in such a way that twice its abscissa always exceeds the ordinate by 4.

2. Find the equation to the locus of a point which is always equidistant from the two points $(0, 0)$ and $(3, 4)$.

3. Find the locus of a point which moves in such a way that its distance from the point $(3, 0)$ is always equal to its distance from the origin.

4. Find the equation to the locus of a point which moves in such a way that twice its distance from the x-axis always exceeds three times its distance from the y-axis by 4.

5. Find the equation to the locus of a point which moves so that its distance from the y-axis is double its distance from the point $(2, 2)$. [C. U.]

6. A point moves so that its distance from the point $(1, 0)$ is always equal to its distance from the axis of y. Find the equation to its locus.

7. Find the equation to the locus of a point which moves in such a way that its distance from $(5, 12)$ is always equal to 13.

8. A point moves in such a way that the sum of the squares of its distances from the two fixed points $(a, 0)$ and $(-a, 0)$ is constant and is equal to $2k^2$. Find the equation to its locus.

9. The co-ordinates of two vertices of a triangle are $(3, 2)$ and $(5, 6)$. Find the equation to the locus of the third vertex if the area of the triangle be 12 sq. units.

10. Find the equation to the locus of a point which moves in a plane so that the sum of its distances from two fixed straight lines at right angles to each other, is always equal to a constant quantity k.

11. A moving line passes through a fixed point (a, b) and meets the co-ordinate axes in P and Q. Find the locus of the middle point of PQ.

THE STRAIGHT LINE

Equation of a straight line

10. To find the equation of a straight line parallel to one of the co-ordinate axes.

Suppose QR to be a str. line parallel to the y-axis [fig. 14]. Let it cut the x-axis in R and let $OR = a$ units. Then the distance of QR from y-axis is equal to a units.

Let (x, y) be the co-ordinates of any point F on QR. Now, in any position of F, whatever may be its ordinate, its abscissa will always be equal to a. Even if QR is produced both ways, the abscissa of any point on it will be equal to a, and the abscissa of no point outside it can be equal to a.

Hence, the **equation** of the str. line QR is $\mathbf{x = a}$.

Similarly it can be proved that if TN be a str. line parallel to the x-axis at a distance of b units from it, the ordinate of any point on it (TN) is always equal to b.

So, the equation of TN will be $\boldsymbol{y = b}$.

Corollary : (1) If in the first equation $a = 0$, then the distance of QR from the y-axis being zero, QR must coincide with the y-axis and then $x = 0$.

$\therefore$ **the equation of the y-axis is** $\mathbf{x = 0}$.

(ii) Similarly, if in the second equation $b = 0$, the distance of TN from the x-axis being zero, TN will coincide with the x-axis and y will be equal to 0.

Hence the **equation of the x-axis is** $\mathbf{y = 0}$.

11. Gardient of a straight line.

Let OX, OY be two axes perpendicular to each other and P, P_1, P_2 be three positions of the point P on the same str. line. Let PN, P_1M_1 and P_2M_2 be perpendiculars on the x-axis. Let PR be perpendicular to P_2M_2 from P, and let it cut P_1M_1 at Q. Suppose the str. line PP_1 makes an angle θ with the x-axis.

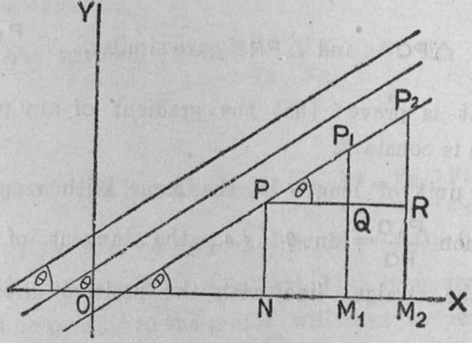


Fig. 13

Now, PN, P_1M_1 and P_2M_2 are the ordinates of the pts. P, P_1 and P_2 respectively and since PR ∥ OX, the $\angle P_2PR = \theta$.

But by the **gradient** or **slope** of a straight line is meant the increase in the ordinate of any point on it corresponding to unit increase in its abscissa. In the figure, when P moves to the position P_1 from the position P, its abscissa increases by the length NM_1 and its ordinate increases by P_1Q. So the increase of ordinate is P_1Q corresponding to the increase NM_1 of the abscissa.

When abscissa increases by 1 unit, the increase of the ordinate$=\dfrac{P_1Q}{NM_1}=\dfrac{P_1Q}{PQ}$ $[\because PQ=NM_1]$ which is the **gradient** of the straight line.

Again, if P moves from the position P to P_2, then the abscissa increases by NM_2 or PR and the ordinate increases by P_2R.

$\therefore$ When the abscissa increases by 1 unit, the increase of the ordinate $= \dfrac{P_2R}{PR}$.

Now, $\because$ $\triangle PQP_1$ and $\triangle PRP_2$ are similar, $\therefore$ $\dfrac{P_1Q}{PQ} = \dfrac{P_2R}{PR}$.

Hence it is proved that the gradient of any point on the straight line is constant.

Now, if unit of length be the same with respect to the two axes, then $\dfrac{P_1Q}{PQ} = \tan\theta$ (*i.e.*, the tangent of the angle made by the straight line with the positive direction of the x-axis.)

Definition of gradient : The gradient or slope of a straight line is the tangent of the angle that it makes with the positive direction of the x-axis.

Gradient is denoted by m.

1. (i) To find the gradient of the straight line joining two points.

Let P and P_1 be two given points [draw fig. 13] and let their co-ordinates be (x_1, y_1) and (x_2, y_2) respectively.

Suppose PP_1 makes an angle θ with the positive direction of the x-axis.

$\therefore$ the **gradient** required $= \tan\theta = \dfrac{P_1Q}{PQ} = \dfrac{y_2 - y_1}{x_2 - x_1}$ or $\dfrac{y_1 - y_2}{x_1 - x_2}$.

[**N. B.** It is seen in fig. 13 that parallel straight lines have the same gradient.]

11. (ii) We have considered before the condition for collinearity of three points.

Here, suppose $A(x_1, y_1)$, $B\ (x_2, y_2)$ and $C\ (x_3, y_3)$ are three points. If the gradient of the straight line AB be equal to the gradient of BC, then as AB and BC have the same inclination with the x-axis, the points A, B and C must be in the same straight line.

Now, the gradient of AB$=\dfrac{y_1-y_2}{x_1-x_2}$ and the gradient of BC$=\dfrac{y_2-y_3}{x_2-x_3}$.

$\therefore$ A, B, C will be collinear, if $\dfrac{y_1-y_2}{x_1-x_2}=\dfrac{y_2-y_3}{x_2-x_3}$,

i.e., if $x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0$.

(iii) By the angle θ (in Art. 11) it is meant that any straight line parallel to the x-axis will have to revolve in the positive direction by angle θ to coincide with the given straight line.

If the position of the given straight line be the same as that of AB in art. 16, then $m=\tan$ XAB (and not tan BAO). Here m being the tangent of an obtuse angle will be negative.

12. To find the equation to the straight line which is inclined to the x-axis at a given angle and cuts off a given intercept from the y-axis.

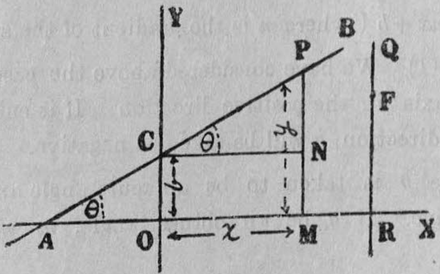


Fig. 14

Suppose the st. line AB is inclined to the x-axis (to its positive direction) at an angle θ and cuts off the intercept OC from the y-axis. Let OC$=b$ units of length.

Take any pt. P on AB and let its co-ordinates be (x, y). Draw PM perpendicular to the x-axis and $CN \perp PM$.

Now, $\angle PCN = \theta$, $CN = OM = x$, $PM = y$,

$NM = CO = b$ and $\tan \theta = \dfrac{PN}{CN}$.

We have $y = PN + NM = PN + b = CN \times \dfrac{PN}{CN} + b = x \tan \theta + b$.

$\therefore \quad y = mx + b$ (where $m = \tan \theta$).

This relation is satisfied by the co-ordinates (x, y) of any point P on AB, but not by the co-ordinates of any point outside AB.

Hence the equation of the st. line AB is $\boldsymbol{y = mx + b}$.

[**Otherwise**] : (Draw fig. 14) suppose AB cuts the y-axis at C and cuts off the intercept $OC(= b)$ from it. Let the gradient of AB be m.

Take any pt. P on AB and let its co-ordinates be (x, y).

Then the co-ordinates of C are $(0, b)$.

$\therefore$ Gradient $m = \dfrac{y - b}{x - 0}$, or, $y - b = mx$,

$\therefore \quad y = mx + b$ (where m is the gradient of the st. line).

[**N. B.** (1) We have considered above the case where AB cuts the y-axis in the positive direction. If it cuts the axis in the negative direction, b will be taken as negative.

(2) Here θ is taken to be an acute angle and m or $\tan \theta$ to be positive. If θ be an obtuse angle, m will be taken negative.

(3) The above equation is called the **tangent form** of the straight line.]

Corollary. (i) If AB passes through the origin, then the intercept cut off by it from the y-axis will be zero, *i.e.*,

$b=0$. In that case the equation becomes $y=mx$. So, the equation of a st. line passing through the origin is $\boldsymbol{y=mx}$.

(ii) If $m=0$ in the equation $y=mx+b$, then the equation will be parallel to the x-axis, for in that case $\tan\theta=0$ and consequently $\theta=0$. So the equation of a st. line parallel to the x-axis is $\boldsymbol{y=b}$.

(iii) If two st. lines have the same gradient (m), they will be equally inclined to the x-axis and consequently they will be parallel to each other.

13. *To find the equation of a straight line passing through a given point and inclined at a given angle to the x-axis.*

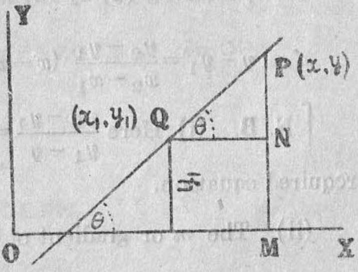


Fig. 15

Suppose the st. line passes through the given point Q (x_1, y_1) and is inclined at an angle θ to the x-axis. From P draw PM perpendicular to the x-axis and let QN⊥PM.

Here PN = PM − NM

$$= y - y_1,$$

$$\text{QN} = x - x_1.$$

Now, $\tan\theta = \dfrac{\text{PN}}{\text{QN}} = \dfrac{y-y_1}{x-x_1}$. If $\tan\theta = m$, then $m = \dfrac{y-y_1}{x-x_1}$.

Hence, the required equation is $\boldsymbol{y - y_1 = m(x - x_1)}$.

[**Otherwise**] : Suppose the st. line passes through the point (x_1, y_1) and makes an angle θ with the x-axis. Let $y=mx+b$ be the equation of the st. line. Then the equation is satisfied by the co-ordinates (x_1, y_1).

$\therefore\ \ y=mx+b\cdots(1)$ and $y_1=mx_1+b\cdots(2)$ [Putting y_1 for y and x_1 for x ; m and b are constant]

Now, subtracting (2) from (1) we have

$y-y_1=m(x-x_1)$, which is the required equation.

14. *To find the equation of a straight line passing through two given points.*

Suppose the st. line passes through the given points (x_1, y_1) and (x_2, y_2) and let its equation be $y = mx + b$...(1) where m and b are unknown.

As the two points lie on the st. line, the equation $y = mx + b$ is satisfied by the co-ordinates (x_1, y_1) and (x_2, y_2) of these points.

$\therefore\ y_1 = mx_1 + b \cdots (2)$ and $y_2 = mx_2 + b \cdots (3)$

Subtracting (2) from (3) we have $y_2 - y_1 = m(x_2 - x_1) \cdots (4)$.
and „ (1) „ (2) „ „ $y_1 - y = m(x_1 - x) \cdots (5)$.

Now, dividing (4) by (5) we have $\dfrac{y_2 - y_1}{y_1 - y} = \dfrac{x_2 - x_1}{x_1 - x}$,

$$\therefore\ y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$ which is the required equation.

[**N. B.** (i) Here $\dfrac{y_2 - y_1}{y_1 - y} = \dfrac{x_2 - x_1}{x_1 - x}$ may be stated as the required equation.

(ii) The m or gradient of this st. line $= \dfrac{y_1 - y_2}{x_1 - x_2}$

$$= \frac{\text{difference of the ordinates}}{\text{difference of the abscissæ}}.\ ]$$

15. *To find the equation of the straight line which cuts off given intercepts from the axes of co-ordinates.*

Let the st. line AB cut the x-axis at A and the y-axis at B and let the intercepts OA $= a$ units and OB $= b$ units.

Let P be any point on AB and let its co-ordinates be (x, y). Draw PM perpendicular to the x-axis.

Here PM $= y$ and OM $= x$.

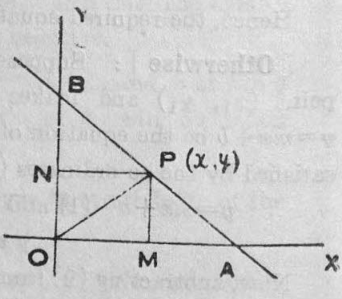


Fig. 16

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Now, $\because$ $\triangle^s \mathsf{APM}$, AOB are similar,

$$\therefore \quad \frac{\mathsf{OM}}{\mathsf{OA}}=\frac{\mathsf{BP}}{\mathsf{AB}},\ i.e.,\ \frac{x}{a}=\frac{\mathsf{BP}}{\mathsf{AB}}\cdots(1)$$

$$\text{and } \frac{\mathsf{PM}}{\mathsf{OB}}=\frac{\mathsf{AP}}{\mathsf{AB}},\ i.e.,\ \frac{y}{b}=\frac{\mathsf{AP}}{\mathsf{AB}}\cdots(2)$$

Adding (1) and (2) we have

$$\frac{x}{a}+\frac{y}{b}=\frac{\mathsf{BP}}{\mathsf{AB}}+\frac{\mathsf{AP}}{\mathsf{AB}}=\frac{\mathsf{BP}+\mathsf{AP}}{\mathsf{AB}}=\frac{\mathsf{AB}}{\mathsf{AB}}=1.$$

Hence, the required equation is $\boldsymbol{\frac{x}{a}+\frac{y}{b}=1}$.

[**Otherwise**] : In fig. 16 draw $\mathsf{PN}\perp\mathsf{OY}$ and join OP.

Here $\mathsf{PN}=\mathsf{OM}=x$.

Now, $\triangle\mathsf{AOB}=\triangle\mathsf{AOP}+\triangle\mathsf{BOP}$,

$\therefore \quad \frac{1}{2}.\ \mathsf{OA}.\ \mathsf{OB}=\frac{1}{2}.\mathsf{OA}.\mathsf{PM}+\frac{1}{2}.\ \mathsf{OB}.\mathsf{PN}$,

or, $\frac{1}{2}ab=\frac{1}{2}ay+\frac{1}{2}bx$, or, $ab=ay+bx$,

$\therefore \quad 1=\frac{y}{b}+\frac{x}{a}$ [dividing by ab], *i.e.*, $\frac{x}{a}+\frac{y}{b}=1$.

[**N. B.** (1) We have considered above the case in which the st. line cuts OX and OY in their positive directions. But it may cut one of them in the positive direction and the other in the negative direction *or* it may cut both in their negative directions. In these cases a and b should be taken with their signs (positive or negative).

(2) The equation obtained above is called the **intercept form** of the st. line.]

✓**16.** *To find the equation of a st. line in terms of the perpendicular drawn to it from the origin and the angle that the perpendicular makes with the axis of x.*

Let OR be perpendicular to the st. line AB from the origin O. OR makes an angle α with the positive direction of the x-axis and $OR=p$ units.

To find the equation of the st. line AB.

Take any pt. P (x, y) on AB and from P draw PN perpendicular to the x-axis. Draw $NL \perp OR$ and $PM \perp NL$.

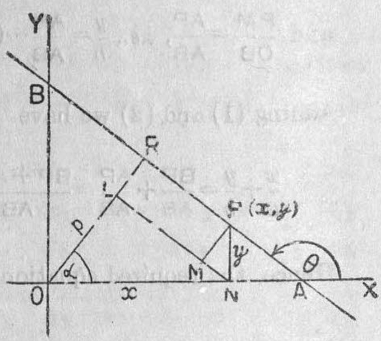


Fig. 17

Now, $OR=OL+LR=OL+PM$. $\because$ OLN is a right-angled triangle, $\therefore\ \alpha+\angle LNO=90^\circ=\angle LNO+\angle PNM$.

$\therefore\ \angle PNM=\alpha$, $ON=x$ and $PN=y$.

$\therefore\ \dfrac{OL}{ON}=\cos\alpha$, $\therefore\ OL=ON\cos\alpha=x\cos\alpha$;

and $\dfrac{PM}{PN}=\sin\alpha$, $\therefore\ PM=PN\sin\alpha=y\sin\alpha$.

$\therefore\ OR=OL+PM$, or, $p=x\cos\alpha+y\sin\alpha$.

Hence, the required equation is $\boldsymbol{x\cos\alpha+y\sin\alpha=p}$.

[**N. B.** This form of the equation is called the **perpendicular** form or canonical **form.**]

[**Another Proof**] Let OP be drawn perpendicular from the origin O to the st. line AB and let it make angle θ with the x-axis. Let AB cut OX at A and OY at B, and let $OP = p$ units. Here the intercepts of the axes are OA and OB.

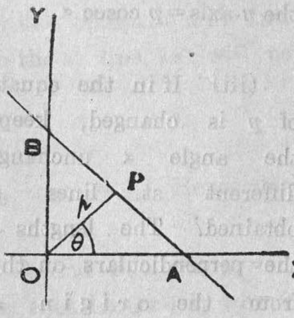


Fig. 18

So, from the equation $\frac{x}{a} + \frac{y}{b} = 1$, in the intercept form we have

$$\frac{x}{OA} + \frac{y}{OB} = 1 \cdots (1).$$

Again, $\frac{OP}{OA} = \cos\theta$, $\therefore$ $OA = \frac{OP}{\cos\theta} = \frac{p}{\cos\theta}$;

and $\frac{OP}{OB} = \cos(90^\circ - \theta) = \sin\theta$, $\therefore$ $OB = \frac{OP}{\sin\theta} = \frac{p}{\sin\theta} \cdots (2)$.

Now, from (1) we have $\dfrac{x}{\frac{p}{\cos\theta}} + \dfrac{y}{\frac{p}{\sin\theta}} = 1$,

or, $\frac{x\cos\theta}{p} + \frac{y\sin\theta}{p} = 1$.

$\therefore$ $x\cos\theta + y\sin\theta = p$, which is the required equation.

Corollary : (i) Putting the equation in the *gradient form* we have $y = (-\cot\alpha)\, x + p\operatorname{cosec}\alpha$.

Here, we find that the gradient m of the st. line $= -\cot\alpha$ *i.e.*, $\tan\theta = -\cot\alpha = \tan(90^\circ + \alpha)$.

$\therefore$ $\theta = 90^\circ + \alpha$, which is evident from the figure also.

(ii) Writing the equation in the *intercept form* we hav

$$\frac{x\cos\alpha}{p} + \frac{y\sin\alpha}{p} = 1, \text{ or, } \frac{x}{p\sec\alpha} + \frac{y}{p\operatorname{cosec}\alpha} = 1.$$

Here the intercept from the x-axis $= p \sec \alpha$ and that from the y-axis $= p \operatorname{cosec} \alpha$.

(iii) If in the equation $x \cos \alpha + y \sin \alpha = p$, the value of p is changed, keeping the angle α unchanged, different st. lines are obtained. The lengths of the perpendiculars on them from the origin are different, but as α is fixed, the same st. line is perpendicular to all these st. lines.

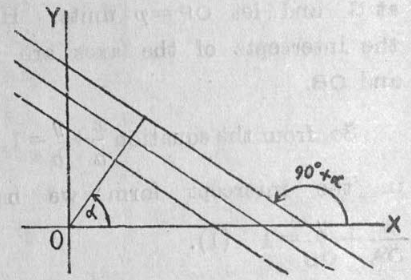


Fig. 19

Hence, the st. lines are parallel to each other.

(iv) Again, if the value of α is changed, while the length (p) of the perpendicular from the origin remains the same, the perpendicular will revolve as α changes.

Hence the foot A of the perpendicular will lie on the circumference of a circle, whose centre is the origin O and radius $= p$.

So, in the different positions of the perpendicular, the st. lines are tangents to the circle. Hence the equation represents some st. lines each of which is a tangent to a fixed circle, having the origin as its centre and the length of the perpendicular from the origin to the st. line as its radius.

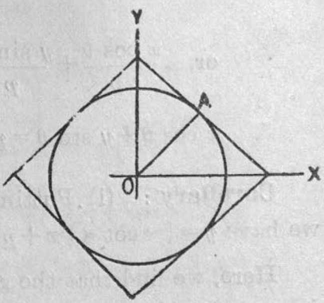


Fig. 20

17. The equation of any straight line is linear.

Let P (x_1, y_1) and Q (x_2, y_2) be any two points on a st. line. Now, any other point A (x, y) will lie on the st. line, *i.e.*, will be collinear with P and Q, if

$$x(y_1 - y_2)+x_1(y_2 - y)+x_2(y - y_1)=0,$$

i.e., if $x(y_1 - y_2)+y(x_2 - x_1)+x_1y_2 - x_2y_1=0$. Hence this will be the equation of that st. line.

Now, putting $y_1 - y_2=a$, $x_2 - x_1=b$ and $x_1y_2 - x_2y_1=c$, the above equation is reduced to the equation $ax+by+c=0$ and this is the **general form** of the first degree equation in x and y.

18. Any linear equation (*i.e.*, a first degree equation in x and y) **represents a straight line.**

The general form of the first degree equation in x and y is $ax+by+c=0$ (where a and b are not zero.)$\cdots$(1).

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be any three points on the locus of the above equation. So, the equation is satisfied by the above co-ordinates.

$$\therefore \quad ax_1+by_1+c=0\cdots\cdots(2)$$
$$ax_2+by_2+c=0\cdots\cdots(3)$$
$$ax_3+by_3+c=0\cdots\cdots(4)$$

From (2) and (3) we have by the method of cross-multiplication, $\dfrac{a}{y_1 - y_2}=\dfrac{b}{x_2 - x_1}=\dfrac{c}{x_1y_2 - x_2y_1}=k$ (Suppose)

Now, putting the values of a, b, c in (4) we have

$$k\{x_3(y_1 - y_2)+y_3(x_2 - x_1)+(x_1y_2 - x_2y_1)\}=0,$$

$$\therefore \quad x_3(y_1 - y_2)+y_3(x_2 - x_1)+(x_1y_2 - x_2y_1)=0,$$

or, $(x_1y_2 - x_2y_1)+(x_2y_3 - x_3y_2)+(x_3y_1 - x_1y_3)=0.$

Hence the pts. (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear.

$\therefore \quad ax+by+c=0$ represents a straight line.

Suppose P to be a point on the same plane.

Let the co-ordinates of P, referred to the original axes, be (x, y) and referred to the new axes be (x_1, y_1). Draw PM and O′N pependicular to OX and let PM cut O′X′ at M′.

Now, we have

$\mathrm{OM}=x$, $\mathrm{MP}=y$, $\mathrm{O'M'}=x_1$, $\mathrm{M'P}=y_1$, $\mathrm{ON}=x'$ and $\mathrm{NO'}=y'$.

$\therefore \quad x=\mathrm{OM}=\mathrm{ON}+\mathrm{NM}=\mathrm{ON}+\mathrm{O'M'}=x'+x_1$;

and $y=\mathrm{MP}=\mathrm{MM'}+\mathrm{M'P}=\mathrm{NO'}+\mathrm{M'P}=y'+y_1$.

Hence we find that the origin is transferred to the point (x', y'), when we substitute $x'+x_1$ and $y'+y_1$ for the co-ordinates x and y respectively.

This is true for the rectangular as well as the oblique axes.

Examples (3)

Ex. 1. Find the equation of the st. line parallel to the x-aixs and passing through the point (4, 7).

The equation of the st. line parallel to the x-axis is $y=b$.

As the st. line passes through the pt. (4, 7), we have $b=7$.

$\therefore$ the required equation is $y=7$.

Ex. 2. Find the equation of the line that passes through the point $(-1, 4)$ and has a gradient 2.

We know that if the gradient of the st. line passing through the pt. (x_1, y_1) be m, its equation is $y-y_1=m(x-x_1)$.

Here $m=2$, $x_1=-1$, $y_1=4$.

$\therefore$ the required equation is $y-4=2\{x-(-1)\}$

or, $y-4=2x+2$, or, $2x-y+6=0$.

Ex. 3. Find the gradient of the line joining the points (2, 4) and (1, 2).

The gradient of the st. line passing through the points (x_1, y_1) and $(x_2, y_2)=\dfrac{y_1-y_2}{x_1-x_2}$.

Here $y_1=4$, $y_2=2$ and $x_1=2$, $x_2=1$.

$\therefore$ the required gradient $=\dfrac{4-2}{2-1}=2$.

Ex. 4. Find the acute angle between the two st. lines whose gradients are 1 and $\dfrac{1}{\sqrt{3}}$.

If m_1 and m_2 be the gradients of two st. lines and if θ be the angle between them, then we have $\tan\theta=\dfrac{m_1-m_2}{1+m_1m_2}$.

$$\therefore \quad \text{Here } \tan\theta=\frac{1-\frac{1}{\sqrt{3}}}{1+1\times\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{(\sqrt{3}-1)^2}{3-1}$$
$$=2-\sqrt{3}=\tan 15^\circ.$$

$\therefore$ $\theta=15^\circ$. $\therefore$ the required acute angle $=15^\circ$.

Ex. 5. Find the equation of the straight line cutting off an intercept 2 units from the negative side of the y-axis and inclined at 120° to the x-axis.

Let the equation be $y=mx+c$.

Here $m=\tan 120^\circ=\tan(180^\circ-60^\circ)=-\tan 60^\circ=-\sqrt{3}$, and $c=-2$ ($\because$ the intercept here is 2 units on the negative side of the y-axis.)

$\therefore$ the required equation is

$$y=x(-\sqrt{3})-2, \quad \text{or,} \quad y+x\sqrt{3}+2=0.$$

Ex. 6. Find the equation of the line passing through the points $(-1, 2)$ and $(3, -4)$.

The equation of the st. line joining the points (x_1, y_1) and (x_2, y_2) is $\dfrac{y-y_1}{x-x_1}=\dfrac{y_2-y_1}{x_2-x_1}$.

Here $x_1=-1$, $x_2=3$, and $y_1=2$, $y_2=-4$.

$\therefore$ we have $\dfrac{y-2}{x-(-1)}=\dfrac{-4-2}{3-(-1)}$, or, $\dfrac{y-2}{x+1}=\dfrac{-6}{4}=\dfrac{-3}{2}$.

$\therefore$ the required equation is $3x+2y-1=0$.

[**Otherwise**] Let $y=mx+c$ be the required equation.

$\because$ the given points lie on that st. line,

$\therefore$ the equation will be satisfied by the co-ordinates of both the points.

$\therefore$ we have $2=-m+c\cdots(1)$ and $-4=3m+c\cdots(2)$

Solving (1) and (2) we have $m=-\frac{3}{2}$ and $c=\frac{1}{2}$.

$\therefore$ the reqd. equation is $y=-\frac{3}{2}x+\frac{1}{2}$, *i.e.*, $3x+2y-1=0$.

Ex. 7. Find the points at which the line $2y-4x-7=0$ cuts the axes of co-ordinates and find its gradient.

The ordinate of the point of intersection of the st. line and the x-axis must be zero, *i.e.*, there $y=0$.

Putting $y=0$ in the given equation we have

$-4x=7$, or, $x=-\frac{7}{4}$. $\therefore$ the st. line cuts the x-axis at the pt. $(-\frac{7}{4}, 0)$.

Similarly putting $x=0$ in the given equation we have $y=\frac{7}{2}$.

$\therefore$ the st. line cuts the y-axis at the pt. $(0, \frac{7}{2})$.

Again, here the equation is $2y=4x+7$ or $y=2x+\frac{7}{2}$,

$\therefore$ the required gradient $=2$.

Ex. 8. Obtain the equation of the st. line which makes intercepts 2 and 1 on the co-ordinate axes. Also find the intercepts made on the axes by the line $2x+y=5$. [C. U. '44]

Let the equation be $\frac{x}{a}+\frac{y}{b}=1$. Here $a=2$ and $b=1$.

$\therefore$ We have $\frac{x}{2}+\frac{y}{1}=1$, or, $x+2y=2$, which is the required equation.

Again, the equation $2x+y=5$ can be written as $\frac{x}{\frac{5}{2}}+\frac{y}{5}=1$.

$\therefore$ The required intercepts on the axes of x and y are $\frac{5}{2}$ and 5 respectively.

Ex. 9. A straight line passes through the point $(2, \frac{5}{3})$ and is such that the portion of it intercepted between the axes is divided at the point internally in the ratio 5 : 4. Find its equation.

Let the equation be $\frac{x}{a}+\frac{y}{b}=1$.

Suppose it cuts the x-axis at A and the y-axis at B. Then the co-ordinates of A and B are $(a, 0)$ and $(0, b)$ respectively.

The co-ordinates of the point at which AB is divided in the ratio $5:4=\left(\frac{5.0+4.a}{5+4}, \frac{5.b+4.0}{5+4}\right)=\left(\frac{4}{9}a, \frac{5}{9}b\right)$; but by the given condition the co-ordinates of the pt. are $(2, \frac{5}{3})$.

$\therefore$ we have $\frac{4}{9}a=2$ or $a=\frac{9}{2}$, and $\frac{5}{9}b=\frac{5}{3}$ or $b=3$.

Hence the reqd. equation is $\frac{x}{\frac{9}{2}}+\frac{y}{3}=1$, or, $2x+3y-9=0$.

Ex. 10. Find the points at which the line $2y+4x+7=0$ cuts the axes of co-ordinates and find its gradient. Also, find the area of the triangle which this line forms with the co-ordinate axes.

The given equation, when reduced to the form $y=mx+c$, becomes $y=-2x-\frac{7}{2}$. $\therefore$ the required gradient $=-2$.

Again, when reduced to the form $\frac{x}{a}+\frac{y}{b}=1$, it becomes $\frac{x}{-\frac{7}{4}}+\frac{y}{-\frac{7}{2}}=1$.

$\therefore$ The co-ordinates of the points, where it cuts the x-axis and the y-axis are respectively $(-\frac{7}{4}, 0)$ and $(0, -\frac{7}{2})$.

The triangle formed by the st. line with the axes is a right-angled triangle. The length of the side along the x-axis is $\frac{7}{4}$ and that along the y-axis is $\frac{7}{2}$.

$\therefore$ The area of the $\Delta=\frac{1}{2}\times\frac{7}{4}\times\frac{7}{2}=\frac{49}{16}$ or $3\frac{1}{16}$ sq. units.

Ex. 11. Find the equation of the line which makes equal intercepts on the axes and passes through $(3, -5)$.

Let the equation of the st. line be $\frac{x}{a}+\frac{y}{b}=1$.

Here the intercepts being equal, we have $a=b$.

$\therefore$ the equation is $\frac{x}{a}+\frac{y}{a}=1$ or $x+y=a$.

Again, as the st. line passes through the pt. $(3, -5)$,

we have $3-5=a$ or $a=-2$.

Hence, the reqd. equation is $x+y=-2$ or $x+y+2=0$.

Ex. 12. Find the equation of the line passing through $(2, 3)$ and parallel to the join of $(4, -5)$ and $(-7, 3)$.

The gradient of the st. line passing through the pts. $(4, -5)$ and $(-7, 3)=\dfrac{-5-3}{4-(-7)}=-\dfrac{8}{11}$.

The equation of the st. line passing through the pt. $(2, 3)$ is $y-3=m(x-2)$.

$\because$ this st. line is parallel to the st. line passing through the pts. $(4, -5)$ and $(-7, 3)$,

$\therefore$ the gradients of both the st. lines must be equal.

$\therefore$ $m=-\frac{8}{11}$.

Hence, ths reqd. equation is $y-3=-\frac{8}{11}(x-2)$

or, $8x+11y-49=0$.

Ex. 13. A straight line cuts off intercepts 7 and $5\frac{1}{4}$ from the axes ; find its equation and determine the ratio in which the join of the points $(-9, 5)$ and $(7, 9)$ is divided by this line.

As the intercepts from the axes are 7 and $5\frac{1}{4}$, the equation of the st. line is $\dfrac{x}{7}+\dfrac{y}{5\frac{1}{4}}=1$, or, $3x+4y=21$.

Suppose this st. line cuts the join of the pts. $(-9, 5)$ and $(7, 9)$ in the ratio $m:n$.

$\therefore$ The co-ordinates of the point of section are

$$\left(\frac{7m-9n}{m+n}, \frac{9m+5n}{m+n}\right).$$

$\because$ this point of section also lies on the st. line $3x+4y=21$,

$\therefore$ we have $3\left(\dfrac{7m-9n}{m+n}\right)+4\left(\dfrac{9m+5n}{m+n}\right)=21$,

or, $3(7m-9n)+4(9m+5n)=21(m+n)$, or, $36m=28n$,

$\therefore$ $\dfrac{m}{n}=\dfrac{28}{36}=\dfrac{7}{9}$. $\therefore$ the required ratio is **7 : 9**.

Ex. 14. A straight line forms a right-angled triangle with the axes of co-ordinates. If the hypotenuse is 13 and the area of the triangle is 30, find the equation of the straight line.

[C. U. '38]

Let the equation of the st. line be $\dfrac{x}{a}+\dfrac{y}{b}=1$.

It cuts the axes at the points $(a, 0)$ and $(0, b)$,

$\therefore$ the length of its intercept between the axes $=\sqrt{a^2+b^2}$,

and by the problem this is the hypotenuse of the triangle.

$\therefore$ $\sqrt{a^2+b^2}=13\ldots(1)$

Again, the area of the triangle $=\frac{1}{2}ab$.

$\therefore$ $\frac{1}{2}ab=30$, $\therefore$ $ab=60\cdots(2)$.

From (1) and (2) we have

$$\left.\begin{aligned} a^2+b^2+2ab&=(13)^2+2\times60=289\\ \text{and } a^2+b^2-2ab&=(13)^2-2\times60=\ \ 49\end{aligned}\right\}$$

or, $(a+b)^2=289$ and $(a-b)^2=49$,

$$\left.\begin{aligned}\therefore\ a+b&=\pm17\\ \text{and } a-b&=\pm\ 7\end{aligned}\right\}\quad \therefore\ \begin{aligned}&a=\pm12 \text{ and } b=\pm5 \text{ ; and}\\ &a=\pm5 \text{ and } b=\pm12.\end{aligned}$$

Hence, the required equation is

$$\frac{x}{\pm12}+\frac{y}{\pm5}=1, \textit{ i.e.}, 5x+12y=\pm60,$$

and $\dfrac{x}{\pm5}+\dfrac{y}{\pm12}=1$, *i.e.*, $12x+5y=\pm60$.

Ex. 19. If p and p_1 be the perpendiculars from the origin upon the lines $x \sin\theta + y \cos\theta = \frac{a}{2}\sin 2\theta$ and $x \cos\theta - y \sin\theta = a \cos 2\theta$, prove that $4p^2 + p_1{}^2 = a^2$. [C. U. '28, '58]

Here, $p = \dfrac{\frac{a}{2}\sin 2\theta}{\sqrt{\sin^2\theta + \cos^2\theta}} = \frac{a}{2}\sin 2\theta$, $\therefore$ $2p = a \sin 2\theta$.

Again, $p_1 = \dfrac{a \cos 2\theta}{\sqrt{\cos^2\theta + \sin^2\theta}} = a \cos 2\theta$.

$$\therefore \quad 4p^2 + p_1{}^2 = (2p)^2 + (p_1)^2 = a^2 \sin^2 2\theta + a^2 \cos^2 2\theta$$
$$= a^2 (\sin^2 2\theta + \cos^2 2\theta) = a^2 \times 1 = a^2.$$

Ex. 20. Transform to parallel axes through the point $(-3, 2)$ the equations :

(i) $3x + 2y - 7 = 0$ and (ii) $2x^2 + y^2 + 4x - 4y = 0$.

Putting $x = x' - 3$ and $y = y' + 2$ in (i) and (ii) we have

(i) $3(x' - 3) + 2(y' + 2) - 7 = 0$,

or, $3x' + 2y' - 12 = 0$.

And (ii) $2(x' - 3)^2 + (y' + 2)^2 + 4(x' - 3) - 4(y' + 2) = 0$,

or, $2(x'^2 - 6x' + 9) + (y'^2 + 4y' + 4) + 4x' - 12 - 4y' - 8 = 0$,

or, $2x'^2 + y'^2 - 8x' + 2 = 0$.

Exercise 3

1. State the gradient of the lines passing through the following pairs of points :—

(i) $(1, -2)$ and $(3, 4)$; (ii) $(-5, 3)$ and $(9, 5)$;
(iii) $(a, b - a)$ and $(a + b, b)$; (iv) $(0, -5)$ and $(-4, 7)$;
(v) $(8, 3)$ and $(-2, 3)$.

2. Find the gradient of the following lines and also the co-ordinates of the points on the axis of x through which these lines pass :—

(i) $3x + 2y = 9$; (ii) $y - 3x = 6$; (iii) $x + y = 0$;
(iv) $2x - y + 5 = 0$; (v) $x \operatorname{cosec}\theta + y \sec\theta + r = 0$.

3. Find the angles at which the lines joining the following pairs of points are inclined to the axis of x and also the co-ordinates of the points on the axis of y through which these lines pass :—

(i) $(0, -3)$ and $(0, 5)$; (ii) $(1, 2)$ and $(3, 4)$;
(iii) $(-6, 1)$ and $(-3, -2)$;
(iv) $\left(-\frac{1}{\sqrt{3}}, 1\right)$ and $(\sqrt{3}, 5)$; (v) $(1, 1)$ and $\left(\frac{2}{3}, \frac{3}{4}\right)$.

4. Find the intercepts on the axes of the following lines :—

(i) $2x - 9y + 6 = 0$; (ii) $3x - 4y + 1 = 0$;
(iii) $y \cos\theta - x \sin\theta = r$.

5. Find the equations to the straight lines cutting off the following intercepts from the axes of x and y respectively :—

(i) 3 and 2 ; (ii) -5 and -4 ; (iii) 3 and $-\frac{9}{5}$;
(iv) $-\frac{13}{5}, \frac{13}{9}$; (v) $\frac{a}{b}$ and 1 ; (vi) $\frac{a}{b}\sec\alpha$ and $\frac{b}{a}\cos\alpha$.

6. Find the equations to the straight lines passing through the following pairs of points :—

(i) $(5, -2)$ and $(-3, 7)$; (ii) $(0, b)$ and $(-a, 0)$;
(iii) $(c\cos C, d\sin C)$ and $(c\cos D, d\sin D)$;
(iv) $(\alpha p_1^2, 2\alpha p_1)$ and $(\alpha p_2^2, 2\alpha p_2)$;
(v) $(h\sec\alpha, k\tan\alpha)$ and $(h\sec\beta, k\tan\beta)$;
(vi) $(3, -4)$ and $(1, 2)$. [U. U. 1948]

7. Find the equations to the sides of the triangles whose vertices are given by :—

(i) $(-4, 3)$, $(7, -3)$ and $(5, 8)$;
(ii) $(-2, 5)$, $(5, -2)$ and $(10, 10)$;
(iii) $(0, 0)$, $(5, -2)$ and $(6, 9)$.

8. Reduce the following equations to the perpendicular form (i. e., $x \cos \alpha + y \sin \alpha = p$) :—

(i) $\sqrt{3}x + y = 8$ (ii) $x - y + 7\sqrt{2} = 0$

(iii) $x + y + 4 = 0$ (iv) $6x - 13y + 19 = 0$.

9. Find the lengths of the perpendiculars from the origin on the lines whose equations are given below :—

(i) $4x + 3y - 5 = 0$; (ii) $5x - 12y = 26$;

(iii) $2x + 3y + 7 = 0$; (iv) $by \cos \theta - ax \sin \theta + ab = 0$.

10. Find the equation to a straight line—

(*a*) which is inclined at 60° to the x-axis and cuts the axis of y at unit distance from the origin.

(*b*) which is inclined at 120° to the axis of x and the length of the perpendicular on the line from the origin is 5.

(*c*) which passes through the point (5, − 7) and makes equal intercepts on the axes of co-ordinates.

(*d*) which is inclined at an angle of 45° to the axis of x and which bisects the join of the points (4, 7) and (6, 5).

(*e*) which passes through the point (5, 6) and has intercepts on the axes equal in magnitude but opposite in sign.

Find also the co-ordinates of the point at which the ordinate is double the abscissa. [C. U.]

✓(*f*) which passes through the point (− 4, 9) and is such that the portion of it intercepted between the axes is divided at the point in the ratio 3 : 2.

$m = 2$
$n = 3$.

✓(*g*) which passes through the point (a, b) and is such that the portion of it intercepted between the axes is bisected at this point.

$\frac{x}{\alpha} + \frac{y}{\beta} = 1$. Since bisect at this point
$\therefore \frac{\alpha}{2} = a$, $\therefore \alpha = 2a$, $\frac{\beta}{2} = b$ $\therefore \beta = 2b$

(*h*) which passes through the point (2, 3) and is parallel to the line joining the points (4, − 7) and (− 7, 4).

$\therefore \frac{x}{2a} + \frac{y}{2b} = 1$

(*i*) which is the perpendicular bisector of the segment joining the points $(2a, 2b)$ and $(2c, 2d)$. [C. U. 1958]

11. Verify that the three points $(1, 5)$, $(3, 14)$ and $(-1, -4)$ are collinear. Also, find the line of collinearity. [C. U. 1957]

12. Find the equations to the straight lines which pass through the origin and trisect the portion of the straight line $4x+3y=12$ intercepted between the axes of co-ordinates. $\frac{x_2-x_1}{x_1-x}$

13. Find the equation of the straight line which passes through the point $(2, 3)$ and is such that the sum of its intercepts on the axes is 10.

14. Given the triangle $A \equiv (8,2)$, $B \equiv (-2,7)$, $C \equiv (-2, -1)$. Find the equation of the median through A. [Mysore, 1946]

Also, prove that the join of the middle points of AB and AC is parallel to BC.

15. A straight line moves such that the sum of the reciprocals of its intercepts on the axes is constant. Prove that the line passes through a fixed point. [Bombay, 1935]

16. Find the equations of the tangents to the circle, whose centre is at the origin and radius equal to 3, at the extremities of a diameter making an angle of 60° with the axis of x.

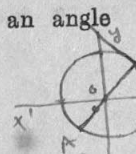
17. Find the equation to the straight line which passes through the point $(5, 6)$ and has intercepts on the axes equal in magnitude but opposite in sign. Find also the co-ordinates of the point at which the ordinate is double the abscissa. [C. U. 1934]

18. Find the equation of the straight line, which cuts off an intercept -3 from the y-axis and is inclined at an angle of 45° to the positive direction of the x-axis.

coordinate of the ends of diameter -

$\frac{OM}{OA} = \cos 60°$ $\therefore OM = 3\cos 60° = \pm\frac{3}{2}$

$\sin 60°$ $+3\sqrt{3}$



Draw a sketch of the straight line and show from geometrical consideration that this line is at right angles to the straight line $x+y=2$. [C. U. 1939]

19. (*a*) Find the inclination of the straight line $4x+5y+3=0$ to the axis of x and find its intercept on the y-axis.

(*b*) The straight line $3x+y-6=0$ cuts the axes at P and Q. Find the distance between P and Q.

(*c*) Find the length of the intercept made on the axes by the line $\frac{x}{2}+3y=1$.

20. Perpendiculars are drawn from the origin to the lines $x+2y-3=0$ and $2x+3y=5$. Find the equation of the straight line joining the feet of the perpendiculars.

21. Txansform to parallel axes through the point $(-2, 1)$ the equations (i) $x^2-4y+4x+8=0$ and (ii) $2x^2+3y^2-4x+3y-8=0$.

22. What will the equation $x^2+y^2-2x+3y+4=0$ become, if the origin is moved to the point $(3, 0)$?

23. What does the equation $(x+a)^2+(y-b)^2=c^2$ become, when it is transferred to parallel axes through the point $(-a+c, b)$?

Straight lines

21. *To find the equation of a straight line in symmetrical form.*

Suppose the straight line passing through a given pt. Q (x_1, y_1) makes an angle θ with the positive direction of x-axis. Take any point P (x, y) on the straight line. Draw PM⊥OX, QN⊥OX, QR⊥PM.

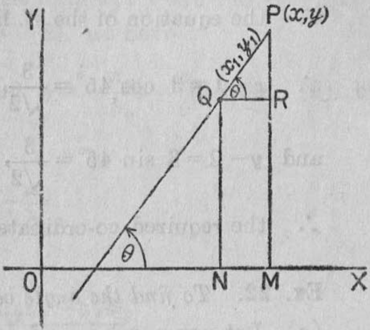


Fig. 23

The equation of the st. line will be

$y - y_1 = \tan\theta(x - x_1)$ [see Art. 13]

$$\text{or,}\quad \frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta}.$$

Suppose PQ $= r$, then QR $= r\cos\theta$,

or, $x - x_1 = r\cos\theta$ [see fig. 23], $\therefore\ \dfrac{x - x_1}{\cos\theta} = r.$

Again, PR $= r\sin\theta$, or, $y - y_1 = r\sin\theta$, $\therefore\ \dfrac{y - y_1}{\sin\theta} = r.$.

$\therefore\ \dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta} = r \cdots (1)$, this is the required equation.

From (1) we have $x = x_1 + r\cos\theta$ and $y = y_1 + r\sin\theta$.

[**N. B.** The special advantage of this form of the equation is (i) that the co-ordinates of a point at a certain distance from a given point on the straight line can be determined, and (ii) also the distance along the line between two given points can be found.]

Example. P is a point $(1, 2)$ and PQ makes an angle of 45° with the x-axis and $PQ=3$. Find the co-ordinates of Q.

Let the co-ordinates of Q be (x, y).

The st. line makes an angle of 45° with the x-axis,

$\therefore$ the equation of the st. line is $\dfrac{x-1}{\cos 45^\circ}=\dfrac{y-2}{\sin 45^\circ}=3,$

$\therefore \quad x-1=3 \cos 45^\circ=\dfrac{3}{\sqrt{2}}, \quad \therefore \quad x=\dfrac{3}{\sqrt{2}}+1,$

and $y-2=3 \sin 45^\circ=\dfrac{3}{\sqrt{2}}, \quad \therefore \quad y=\dfrac{3}{\sqrt{2}}+2.$

$\therefore$ the required co-ordinates are $\left(\dfrac{3}{\sqrt{2}}+1, \dfrac{3}{\sqrt{2}}+2\right)$.

Ex. 22. *To find the angle between two given straight lines.*

(*a*) Let $y=m_1x+c_1$ and $y=m_2x+c_2$ be the equations of the two st. lines and let the angle between them be θ.

If the st. lines make angles α and β with the positive direction of the x-axis, then $\tan \alpha=m_1$ and $\tan \beta=m_2$.

We find from the figure $\theta=\alpha-\beta$.

$$\therefore \quad \tan \theta=\tan(\alpha-\beta)$$

$$=\frac{\tan \alpha-\tan \beta}{1+\tan \alpha \tan \beta}$$

$$=\frac{m_1-m_2}{1+m_1 m_2}.$$

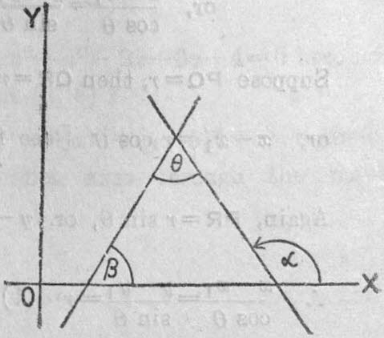


Fig. 24

$\therefore \quad \theta=\tan^{-1}\dfrac{m_1-m_2}{1+m_1m_2}\cdots(1)$, which gives the angle between the straight lines.

[**N. B.** When two st. lines (not at right angles) intersect, one angle between them is acute and the other is obtuse.

If the obtained value of $\tan\theta$ is positive, then it is to be understood that the acute angle is found, and if the value of $\tan\theta$ be negative, then the obtuse angle between the st. lines is found.]

(*b*) If the equations be in the form $a_1x+b_1y+c_1=0\cdots(1)$ and $a_2x+b_2y+c_2=0\cdots(2)$, we have

$y=-\frac{a_1}{b_1}x-\frac{c_1}{b_1}$ from (1) and $y=-\frac{a_2}{b_2}x-\frac{c_2}{b^2}$ from (2) by reducing them to tangent form.

$$\therefore\ m_1=-\frac{a_1}{b_1} \text{ and } m_2=-\frac{a_2}{b_2}.$$

$$\therefore\quad \tan\theta=\frac{-\frac{a_1}{b_1}+\frac{a_2}{b_2}}{1+\frac{a_1}{b_1}\times\frac{a_2}{b^2}}=\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}.$$

$$\therefore\quad \theta=\tan^{-1}\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}\cdots(2).$$

(*c*) Let the equations of the st. lines be

$$x\cos\alpha_1+y\sin\alpha_1=p_1$$
$$\text{and } x\cos\alpha_2+y\sin\alpha_2=p_2.$$

Here the two perpendiculars that will be drawn from the origin to the given st. lines will make angles α_1 and α_2 respectively with the positive direction of the x-axis.

$\therefore$ evidently the angle between the st. lines will be

$$\alpha_1-\alpha_2 \text{ or } \pi-(\alpha_1-\alpha_2),\ i.e.,$$
$$\theta=\alpha_1-\alpha_2 \text{ or } \theta=\pi-(\alpha_1-\alpha_2).$$

23. Condition of parallelism of two straight lines.

If the two st. lines be parallel, α will be equal to β.

$$\therefore\quad \theta=\alpha-\beta=0.\quad \therefore\quad \tan\theta=\tan 0^\circ=0.$$

$$\therefore\quad \text{from } \tan\theta=\frac{m_1-m_2}{1+m_1m_2}, \text{ we have } m_1-m_2=0,$$

$\therefore$ By interchanging the coefficients of x and y in equation-(2) with the sign changed in any one of them and adding a new constant term, we get the equation of the st. line perpendicular to it. Now, if the value of this constant term be found from a second given condition, we obtain the equation required.

As for example, suppose we are to find the equation of the st. line perpendicular to the st. line $2x+3y+1=0$.

Here interchanging the coefficients of x and y, we have $3x$ and $2y$. Now, by changing the sign in one of them and adding a constant term k, we have $3x-2y+k=0$.

Rule : To find the equation of a st. line perpendicular to a given st. line, interchange the coefficients of x and y in the given equation altering the sign in any one of them and add a new constant term.

26. To find the point of intersection of two given straight lines.

Let the equations of the two st. lines be

$a_1x+b_1y+c_1=0 \cdots (1)$ and $a_2x+b_2y+c_2=0 \ldots (2)$.

The point of intersection of the two st. lines is the only point common to them. So the equations will be satisfied by the co-ordinates of the point. Let its co-ordinates be (α, β).

Then we have $a_1\alpha+b_1\beta+c_1=0 \cdots\cdots (3)$

and $a_2\alpha+b_2\beta+c_2=0 \cdots\cdots (4)$

From (3) and (4) we have by the method of cross-multiplication

$$\frac{\alpha}{b_1c_2-b_2c_1}=\frac{\beta}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}.$$

$\therefore$ $\alpha=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}$ and $\beta=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$, these are the required co-ordinates.

[**N. B.** Definite co-ordinates can be obtained, if $a_1b_2 - a_2b_1 \neq 0$. But if $a_1b_2 - a_2b_1 = 0$, the co-ordinates of the pt. of intersection are infinite. We have already seen that if $a_1b_2 - a_2b_1 = 0$, *i.e.*, if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, then the two st. lines will be parallel. Hence parallel lines may be considered as lines whose point of intersection is at an infinite distance.]

27. To find the equation of a straight line passing through the point of intersection of two given straight lines.

Let the equations of the two st. lines be

$a_1x + b_1y + c_1 = 0 \ldots (1)$ and $a_2x + b_2y + c_2 = 0 \ldots (2)$

Now, $a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0 \ldots (3)$ being an equation of the first degree in x and y always represents a st. line. Here k may be any constant and for its different values the equation will represent different straight lines.

The co-ordinates of the pt. of intersection of the straight lines (1) and (2) are $\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$.

These co-ordinates satisfy the equation-(3). Hence the st. line-(3) will pass through the pt. of intersection of the st. lines (1) and (2).

Hence the equation-(3) is the required equation of the st. line passing through the point of intersection of the straight lines (1) and (2).

28. To find the condition of concurrence of three straight lines.

Let the equations of the thres st. lines be

$$a_1x + b_1y + c_1 = 0 \ldots (1)$$
$$a_2x + b_2y + c_2 = 0 \ldots (2)$$
$$\text{and } a_3x + b_3y + c_3 = 0 \ldots (3)$$

Ex. 2. Find the equation of a straight line which passes through the point $(2, -1)$ and is perpendicular to the straight line $3x-2y=5$. [J. B. A]

[First Method] Here the given equation is $3x-2y=5$.

The equation of any st. line perpendicular to it will be

$$2x+3y+k=0.$$

$\because$ this st. line passes through the point $(2, -1)$,

$\therefore$ $2\times2+3\times-1+k=0$, $\therefore$ $k=-1$.

Hence, the required equation is $2x+3y-1=0$.

[Second Method] The equation of a st. line passing through the point $(2, -1)$ is $y+1=m(x-2)\cdots\cdots(1)$

The gradient of the st. line $3x-2y=5\cdots(2)$ is $\frac{3}{2}$.

$\therefore$ the st. line-(1) will be perpendicular to the st. line-(2), if $m\times\frac{3}{2}=-1$ or if $m=-\frac{2}{3}$.

Hence, the required equation is $y+1=-\frac{2}{3}(x-2)$

or, $2x+3y-1=0$.

Ex. 3. Find the equation of a straight line which passes through the point $(-3, -4)$ and is parallel to the straight line $x\cos80^\circ+y\sin80^\circ+7=0$.

The equation of the st. line parallel to the given st. line is $x\cos80^\circ+y\sin80^\circ+k=0$ [(iii) of Art. 16]

$\because$ this st. line passes through the point $(-3, -4)$,

$\therefore$ $-3\cos80^\circ-4\sin80^\circ+k=0$,

or, $k=3\cos80^\circ+4\sin80^\circ$.

$\therefore$ the required equation is

$$x\cos80^\circ+y\sin80^\circ+3\cos80^\circ+4\sin80^\circ=0,$$

or, $(x+3)\cos80^\circ+(y+4)\sin80^\circ=0$.

Ex. 4. Find the equation of a straight line passing through the point $(2, 1)$ and perpendicular to the straight line $x\cos6^\circ+y\sin6^\circ=3$.

The equation of any st. line perpendicular to the st. line $x \cos 6^\circ + y \sin 6^\circ = 3$ will be $x \sin 6^\circ - y \cos 6^\circ + k = 0$.

Since this st. line passes through the point (2, 1), we have $2 \sin 6^\circ - \cos 6^\circ + k = 0$, $\therefore\ k = \cos 6^\circ - 2 \sin 6^\circ$.

Hence the required equation is

$$x \sin 6^\circ - y \cos 6^\circ + \cos 6^\circ - 2 \sin 6^\circ = 0,$$

or, $(x-2) \sin 6^\circ - (y-1) \cos 6^\circ = 0$.

Ex. 5. Find the equation to the perpendicular bisector of the segment joining the points (4, −5) and (−7, 3).

Here, the gradient of the st. line joining the

points $(4, -5)$ and $(-7, 3) = \dfrac{-5-3}{4-(-7)} = -\dfrac{8}{11}$.

$\therefore$ The gradient of the perpendicular to that st. line. is $\frac{11}{8}$.

Again, the co-ordinates of the mid point of the segment joining the points (4, −5) and (−7, 3) are

$$\left(\frac{4-7}{2}, \frac{-5+3}{2}\right) \text{ or } \left(-\frac{3}{2}, -1\right).$$

$\because$ the perpendicular bisector passes through the point $(-\frac{3}{2}, -1)$,

$\therefore$ its equation is $y + 1 = \frac{11}{8}(x + \frac{3}{2})$, or, $16y - 22x - 17 = 0$.

Ex. 6. If the straight line $\dfrac{x}{a} + \dfrac{y}{b} = 1$ passes through the point of intersection of the lines $2x - y = 1$ and $3x - 4y + 6 = 0$ and is parallel to the line $4x + 3y - 6 = 0$, find a and b.

[C. U. 1948]

Solving the equations $2x - y = 1$ and $3x - 4y + 6 = 0$, we have $x = 2$ and $y = 3$.

$\therefore$ these two st. lines intersect at the point (2, 3).

Again, the equation of the st. line parallei to the st. line $4x+3y-6=0$ will be $4x+3y+k=0$.

$\because$ this parallel st. line passes through the point (2, 3),

$\therefore \quad 4\times 2+3\times 3+k=0, \quad \therefore \quad k=-17.$

$\therefore$ the equation of the st. line is $4x+3y-17=0$.

Writing it in the intercept form, we have $\dfrac{x}{\frac{17}{4}}+\dfrac{y}{\frac{17}{3}}=1$.

As this line is identical with the line $\dfrac{x}{a}+\dfrac{y}{b}=1$,

$$\therefore \quad a=\tfrac{17}{4} \text{ and } b=\tfrac{17}{3}.$$

Ex. 7. Find the equation to the straight line which passes through the point (5, 4) and the point of intersection of the lines $2x+3y-1=0$ and $3x-4y+7=0$.

[**First Method**] Solving $2x+3y-1=0$ and $3x-4y+7=0$, we have $x=-1$ and $y=1$. $\therefore$ the two st. lines intersect at the point $(-1, 1)$.

The equation of the st. line passing through the points (5, 4) and $(-1, 1)$ is

$$\frac{y-4}{4-1}=\frac{x-5}{5-(-1)}, \text{ or, } \frac{y-4}{3}=\frac{x-5}{6}, \text{ or, } x-2y+3=0.$$

$\therefore$ the required equation is $x-2y+3=0$.

[**Second Method**] The equation of any st. line, passing through the point of intersection of the st. lines $2x+3y-1=0$ and $3x-4y+7=0$, is

$$2x+3y-1+k(3x-4y+7)=0.$$

$\because$ it passes through the point (5, 4), $\therefore$ we have $2.5+3.4-1+k(3.5-4.4+7)=0$, or, $21+6k=0$, $\therefore$ $k=-\frac{7}{2}$.

$\therefore$ the required equation is $2x+3y-1-\frac{7}{2}(3x-4x+7)=0$,

$$\text{or,} \quad x-2y+3=0.$$

Ex. 8. Find the equation of the line through the point of intersection of the lines $4x+y-4=0$ and $3x+2y-5=0$ and perpendicular to the line $x-2y+1=0$.

[**First Method**] Solving the equations $4x+y-4=0$ and $3x+2y-5=0$, we have $x=\frac{3}{5}$, $y=\frac{8}{5}$. $\therefore$ the co-ordinates of their point of intersection are $(\frac{3}{5}, \frac{8}{5})$.

Now, the equation of any st. line perpendicular to the line $x-2y+1=0$ is $2x+y+k=0$.

$\because$ it passes through the point. $(\frac{3}{5}, \frac{8}{5})$,

$\therefore$ $2\times\frac{3}{5}+\frac{8}{5}+k=0$, $\therefore$ $k=-\frac{14}{5}$.

$\therefore$ the required equation is $2x+y-\frac{14}{5}=0$,

or, $10x+5y-14=0$.

[**Second Method**] The equation of any st. line passing through the point of intersection of the st. lines $4x+y-4=0$ and $3x+2y-5=0$ will be $4x+y-4+k(3x+2y-5)=0$,

or, $(3k+4)x+(2k+1)y-(4+5k)=0\cdots(1)$

Now, the st. line-(1) will be perpendicular to the st. line $x-2y+1=0$, if $(3k+4)\times1+(2k+1)\times-2=0$ [see Art. 24(B)].

i.e., if $k=2$.

$\therefore$ the reqd. equation is $4x+y-4+2(3x+2y-5)=0$,

or, $10x+5y-14=0$.

Ex. 9. (*a*). Find the equation of the straight line passing through the intersection of the lines $x-2y-b=0$ and $x+3y-2b=0$ and parallel to the line $3x+4y=0$.

[**First Method**] Solving $x-2y-b=0$ and $x+3y-2b=0$

we have $x=\dfrac{7b}{5}$ and $y=\dfrac{b}{5}$.

$\therefore$ the co-ordinates of their point of intersection are

$$\left(\frac{7b}{5}, \frac{b}{5}\right).$$

Now, the equation of any st. line parallel to the line $3x+4y=0$ is $3x+4y+k=0$.

$\because$ it passes through the point $\left(\frac{7b}{5}, \frac{b}{5}\right)$,

$\therefore\ 3\times\frac{7b}{5}+4\times\frac{b}{5}+k=0$, or, $5b+k=0$, $\therefore\ k=-5b$.

$\therefore$ the required equation is $3x+4y-5b=0$.

[**Second Method**] The equation of any st. line passing through the point of intersection of the lines $x-2y-b=0$ and $x+3y-2b=0$ is $(x-2y-b)+k(x+3y-2b)=0$,

or, $(k+1)x+y(3k-2)-b(2k+1)=0\cdots(1)$.

Now, the st. line-(1) will be parallel to the st. line $3x+4y=0$, when $\frac{k+1}{3}=\frac{3k-2}{4}$ [vide Art. 23 (B)],

i.e., when $k=2$.

$\therefore$ the required equation is $(2+1)x+(6-2)y-b(4+1)=0$,

or, $3x+4y-5b=0$.

Ex. 9. (*b*). Find the lines through the point of intersection of $y-2x+2=0$ and $y-3x+5=0$, which are at a distance of $\frac{7}{\sqrt{2}}$ from the origin. [U. P. B. 1942]

Solving the given equations we have $x=3$ and $y=4$.

$\therefore$ the co-ordinates of their point of intersection are (3, 4).

The equation of a st. line passing through the point (3, 4) is $y-4=m(x-3)$, or, $mx-y+(4-3m)=0$.

The perpendicular distance of this st. line from the origin $(0,0)=\frac{4-3m}{\sqrt{m^2+(-1)^2}}=\frac{4-3m}{\sqrt{1+m^2}}$. $\therefore$ Here $\frac{4-3m}{\sqrt{1+m^2}}=\frac{7}{\sqrt{2}}$.

or, $\frac{16+9m^2-24m}{1+m^2}=\frac{49}{2}$, or, $31m^2+48m+17=0$,

or, $(31m+17)(m+1)=0$, $\therefore\ m=-1$, or, $-\frac{17}{31}$.

If $m = -1$, the equation is $y - 4 = -1(x-3)$, or, $x+y=7$;

If $m = -\frac{17}{31}$, the equation is $y - 4 = -\frac{17}{31}(x-3)$,

or, $17x+31y=175$.

Hence, the reqd. st. lines are $x+y=7$ and $17x+31y=175$.

Ex. 10. (*a*). Prove that the lines $2x-y+8=0$, $3x+y+2=0$ and $4x+3y-4=0$ are concurrent.

By solving the equations $2x-y+8=0$ and $3x+y+2=0$, the co-ordinates of their point of intersection are found to be $(-2, 4)$.

If the equation $4x+3y-4=0$ be satisfied by the co-ordinates $(-2, 4)$, then the third st. line will also pass through the point of intersection of the first two.

Now, $4x+3y-4=4\times(-2)+3\times 4-4=-8+12-4=0$.

$\therefore$ the third equation is satisfied by the co-ordinates $(-2,4)$.

Hence, the three given lines are concurrent.

Ex. 10. (*b*). Prove that the lines $(b+c)x+ay-d=0$, $(c+a)x+by-d=0$ and $(a+b)x+cy-d=0$ are concurrent.

$$(b+c)x+ay-d=0\cdots(1)$$

$$(c+a)x+by-d=0\cdots(2)$$

$$\text{and}\quad (a+b)x+cy-d=0\cdots(3)$$

The st. lines (1), (2) and (3) will be concurrent, if three constants p, q, r can be found so that

$$p\{(b+c)x+ay-d\}+q\{(c+a)x+by-d\}+r\{(a+b)x+cy-d\}=0\cdots\cdots(A)$$

Here the coefficient of x is $p(b+c)+q(c+a)+r(a+b)\cdots(4)$.

the „ „ y is $pa+qb+rc\cdots(5)$

and the constant term is $(p+q+r)d\cdots(6)$

Now, if such values of p, q, r can be found so that each of (4), (5) and (6) becomes zero, then (1), (2) and (3) will be concurrent.

Evidently (4), (5) and (6) will each be 0, if $p=(b-c)$, $q=(c-a)$ and $r=(a-b)$ and then (A) also will at once be zero.

Hence, st. lines (1), (2) and (3) are concurrent.

Ex. 11. For what value of m will the lines $y=3x-1$, $2y=x+3$ and $3y=mx+4$ be concurrent ? [C. U. '40, '55]

Solving $y=3x-1$ and $2y=x+3$, we have $x=1$, $y=2$.

$\therefore$ the co-ordinates of their point of intersection are (1, 2).

Now, the st. line $3y=mx+4$ will be concurrent with the first two st. lines, if it is satisfied by the co-ordinates (1, 2).

Putting $x=1$, $y=2$ in $3y=mx+4$, we have

$$3\times 2=m\times 1+4, \quad \therefore \quad m=2.$$

Hence, the three lines will be concurrent when $m=2$.

Ex. 12. Find the equation to the straight line which passes through the intersection of the straight lines $3x-4y+1=0$ and $5x+y-1=0$ and cuts off equal intercepts from the axes. [C. U. (B. Sc.) 1947]

The equation of any st. line, which passes through the intersection of the lines $3x-4y+1=0$ and $5x+y-1=0$, is $3x-4y+1+k(5x+y-1)=0$,

$$\text{or,} \quad (5k+3)x+(k-4)y-(k-1)=0 \cdots (1).$$

Reducing it to the intercept form we have

$$\frac{x}{\dfrac{k-1}{5k+3}}+\frac{y}{\dfrac{k-1}{k-4}}=1.$$

$\because$ by the problem the st. line-(1) cuts off equal intercepts from the axes,

$$\therefore \frac{k-1}{5k+3}=\frac{k-1}{k-4}, \text{ or, } (k-1)\left(\frac{1}{5k+3}-\frac{1}{k-4}\right)=0, \therefore k=1 \text{ or } -\tfrac{7}{4}.$$

Now, it is found that if the value $k=1$ be put in the equation-(1), its constant term $k-1$ becomes 0, and consequently the st. line passes through the origin, *i.e.*, it cuts off no intercepts from the axes. So, k cannot be equal to 1 here.

$\therefore$ Putting $k=-\frac{7}{4}$, the required equation will be

$$(5\times -\tfrac{7}{4}+3)x+(-\tfrac{7}{4}-4)y-(-\tfrac{7}{4}-1)=0,$$

or, $-\frac{23}{4}x-\frac{23}{4}y+\frac{11}{4}=0$, *i.e,* $23x+23y-11=0$.

Ex. 13. Find the acute angle between the lines

$$2x+y-3=0 \text{ and } x+3y+2=0.$$

Let θ be the angle between the lines $2x+y-3=0$ and $x+3y+2=0$.

$$\text{We have } \theta=\tan^{-1}\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}$$

[vide formula (2) in Art. 22 (B)]

As the angle here is acute, the values of a_1, b_1, a_2, b_2 should be such that $a_2b_1-a_1b_2$ is positive.

$$\therefore \quad \theta=\tan^{-1}\frac{2\times 3-1\times 1}{2\times 1+(3)\times(1)}=\tan^{-1}1=45^\circ$$

$\therefore$ the required angle is 45°.

Ex. 14. Find the equations to the straight lines which pass through $(3,-2)$ and make an angle of 60° with the line $\sqrt{3}x+y=1$.

[In the fig. 25, AB is the st. line $\sqrt{3}x+y=1$, P is the pt. $(3,-2)$ and each of the st. lines CPD and EPF makes an angle of 60° with AB.

To find the equations of CPD and EPF,

Here you may not draw any figure.]

The equation of a st. line passing through the point $(3,-2)$ is $y+2=m(x-3)\cdots(1)$.

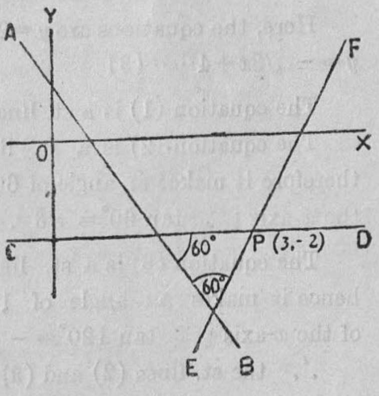


Fig. 25.

$\because$ the line-(1) makes an angle of 60° with the line $\sqrt{3}x+y=1$,

$\therefore$ $\tan 60^\circ=\dfrac{m-(-\sqrt{3})}{1+m(-\sqrt{3})}\cdots$(i) and $\tan 60^\circ=\dfrac{-\sqrt{3}-m}{1+m(-\sqrt{3})}\cdots$(ii)

From (i) we have $\sqrt{3}=\dfrac{m+\sqrt{3}}{1-\sqrt{3}m}$, $\therefore m=0$.

From (ii) we have $\sqrt{3}=\dfrac{-m-\sqrt{3}}{1-\sqrt{3}m}$,

or, $\sqrt{3}-3m=-m-\sqrt{3}$, or, $2m=2\sqrt{3}$, $\therefore m=\sqrt{3}$.

Putting $m=0$ and $m=\sqrt{3}$ in equation-(i) we have $y+2=0$ and $y+2=\sqrt{3}(x-3)$, or, $y-\sqrt{3}x+3\sqrt{3}+2=0$ respectively.

Hence, the required equations are $y+2=0$ and $y-\sqrt{3}x+3\sqrt{3}+2=0$.

[**N. B.** Here m is unknown. It may be greater or less than $-\sqrt{3}$. $\therefore$ in (i) $-\sqrt{3}$ has been subtracted from m and again in (ii) m has been subtracted from $-\sqrt{3}$, and thus two possible st. lines are found.]

Ex. 15. Verify that the three lines $y=2$, $y-\sqrt{3}x=5$, $y+\sqrt{3}x=4$ form an equilateral triangle. Also find the area of the triangle formed. [C. U. 1957]

Here, the equations are $y=2\cdots\cdots(1)$, $y=\sqrt{3}x+5\cdots\cdots(2)$ and $y=-\sqrt{3}x+4\cdots\cdots(3)$

The equation-(1) is a st. line parallel to the x-axis.

The equation-(2) is a st. line whose gradient is $\sqrt{3}$, and therefore it makes an angle of 60° with the positive direction of the x-axis [$\because \tan 60^\circ=\sqrt{3}$].

The equation-(3) is a st. line whose gradient is $-\sqrt{3}$ and hence it makes an angle of 120° with the positive direction of the x-axis [$\because \tan 120^\circ=-\sqrt{3}$].

$\therefore$ the st. lines (2) and (3) make angles of 60° and 120° respectively with the st. line-(1) also, it being parallel to the x-axis.

Hence the st. lines (2) and (3) are inclined to the st. line (1) at an angle of 60° towards each other. So, two of the angles of the triangle formed by the st. lines (1), (2) and (3) are 60° each. $\therefore$ the remaining angle is also 60°. Hence the triangle is equilateral.

[In this example, we may first find the three points of intersection of the st. lines and then showing the lengths of the sides to be equal prove that the triangle is equilateral.]

To find the area of the equilateral triangle, it is sufficient to find the length of one side only, for the area of an equilateral triangle is $\frac{\sqrt{3}}{4}a^2$ (a being the length of its side).

The ordinate of both the points where the st. lines (2) and (3) cut the line $y=2$ is 2. The abscissæ of the points can be found by putting $y=2$ in (2) and (3).

Thus, from (2) we have $x=-\sqrt{3}$ and from (3), $x=\frac{2}{\sqrt{3}}$.

$\therefore$ the length of one side of the triangle

$$= \text{the distance between the two pts.} = \frac{2}{\sqrt{3}}+\sqrt{3}=\frac{5}{\sqrt{3}}.$$

$\therefore$ the area of the triangle $=\frac{\sqrt{3}}{4}\left(\frac{5}{\sqrt{3}}\right)^2=\frac{25}{12}\sqrt{3}$ sq. units.

Ex. 16. Prove analytically that the perpendicular bisectors of the sides of any triangle are concurrent.

Suppose the co-ordinates of the vertices of $\triangle$ABC to be A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3). Let D be the mid point of BC.

Then the co-ordinates of D are $\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$.

The gradient of BC $=\frac{y_2-y_3}{x_2-x_3}$.

$\therefore$ the equation of the perpendicular bisector of BC is

$$y-\frac{y_2+y_3}{2}=-\frac{x_2-x_3}{y_2-y_3}\left(x-\frac{x_2+x_3}{2}\right)$$ [Here the gradient m of the perpendicular on BC $=-\dfrac{1}{\dfrac{y_2-y_3}{x_2-x_3}}$]

or, $2x(x_2-x_3)+2y(y_2-y_3)-(x_2^2-x_3^2)-(y_2^2-y_3^2)=0\cdots(1)$

[simplifying]

Similarly, the equations of the perpendicular bisectors of CA and AB will respectively be

$$2x(x_3-x_1)+2y(y_3-y_1)-(x_3^2-x_1^2)-(y_3^2-y_1^2)=0\cdots\cdots(2)$$

and $2x(x_1-x_2)+2y(y_1-y_2)-(x_1^2-x_2^2)-(y_1^2-y_2^2)=0\cdots(3)$

On addition, the sum of the left hand sides of (1), (2) and (3) becomes 0. [According to Art. 28 (B) here $p=q=r=1$]

$\therefore$ the lines (1), (2) and (3) are concurrent.

Hence, the perpendicular bisectors of the sides of the triangle are concurrent.

Ex. 17. Find the distance from (2, 3) measured along the line $3x-5y+9=0$ up to its point of intersection with $3x-2y=7$.

We have $3x-5y+9=0\cdots(1)$, or, $y=\frac{3}{5}x+\frac{9}{5}$.

The gradient of (1) is $\frac{3}{5}$, *i. e.*, if the st. line-(1) be inclined to the x-axis at an angle θ, then $\tan\theta=\frac{3}{5}$.

$$\therefore\ \frac{\sin\theta}{3}=\frac{\cos\theta}{5},\ \therefore\ \frac{\sin^2\theta}{9}=\frac{\cos^2\theta}{25}=\frac{\sin^2\theta+\cos^2\theta}{9+25}=\frac{1}{34}$$

$$\therefore\ \frac{\sin\theta}{3}=\frac{\cos\theta}{5}=\frac{1}{\sqrt{34}}.\ \therefore\ \sin\theta=\frac{3}{\sqrt{34}}\text{ and }\cos\theta=\frac{5}{\sqrt{34}}.$$

Now, suppose the equation of the st. line passing through (2, 3) is $\dfrac{x-2}{\cos\theta}=\dfrac{y-3}{\sin\theta}=r$. [The distance of the point (2, 3) from P (x, y), the intersection of the line-(1) and the line $3x-2y=7$, is r (Vide Art. 21].

$\therefore \quad x=2+r\cos\theta$ and $y=3+r\sin\theta$,

or, $x=2+\dfrac{5}{\sqrt{34}}r$ and $y=3+\dfrac{3}{\sqrt{34}}r$.

$\because$ the st. line $3x-2y=7$ also passes through P,

$\therefore \quad 3\left(2+\dfrac{5r}{\sqrt{34}}\right)-2\left(3+\dfrac{3r}{\sqrt{34}}\right)=7$,

or, $6+\dfrac{15}{\sqrt{34}}r-6-\dfrac{6}{\sqrt{34}}r=7$,

or, $\dfrac{9}{\sqrt{34}}r=7$, $\therefore\ r=\dfrac{7}{9}\sqrt{34}$, which is the required distance.

[Here the distance can be found otherwise. Find the co-ordinates of the point of intersection by solving the given equations. Then find the distance between the point (2, 3) and the point of intersection.]

✓ **Ex. 18.** Show that the distance of the point (x_0, y_o) from the line $ax+by+c=0$ measured parallel to a line making an angle θ with the x-axis is $-\dfrac{ax_0+by_0+c}{a\cos\theta+b\sin\theta}$.

[C. U. 1954]

The equatien of a line passing through the point (x_0, y_0) and parallel to a st. line which is inclined to the x-axis at an angle θ is $y-y_0=\tan\theta\,(x-x_0)$, or, $\dfrac{x-x_0}{\cos\theta}=\dfrac{y-y_0}{\sin\theta}\ldots(1)$

Now, if r be the distance of the point (x_0, y_0) from the line $ax+by+c=0$ along the st. line-(1), then we have

$$\frac{x-x_0}{\cos\theta}=\frac{y-y_0}{\sin\theta}=r. \quad \therefore \quad x=x_0+r\cos\theta \text{ and } y=y_0+r\sin\theta.$$

$\because$ The point (x_o, y_0) lies on the st. line $ax+by+c=0$ also,

$\therefore \quad a(x_0+r\cos\theta)+b(y_0+r\sin\theta)+c=0,$

or, $r(a\cos\theta+b\sin\theta)=-(ax_0+by_0+c),$

$$\therefore \quad r=-\frac{ax_0+by_o+c}{a\cos\theta+b\sin\theta}.$$

Ex. 19. Verify that the four lines $y=0$, $y+\sqrt{3}(x-8)=0$, $y=2$ and $y-\sqrt{3}x=0$ form a trapezium which is cyclic. Find the co-ordinates of the four vertices and also the area of the trapezium.

The equations are $y=0 \cdots (1)$, $y+\sqrt{3}(x-8)=0 \cdots (2)$, $y=2 \cdots (3)$ and $y-\sqrt{3}x=0 \cdots (4)$.

Hence, the equation-(1) is the x-axis and equation-(3) is a st. line parallel to the x-axis at a distance of 2 units from it.

The gradients of the equations (2) and (4) are $-\sqrt{3}$ and $\sqrt{3}$ respectively. So they make angles 120° and 60° respectively with the x-axis, *i. e.*, with the st. line $y=0$, $\therefore$ they also make angles 120° and 60° with the line $y=2$.

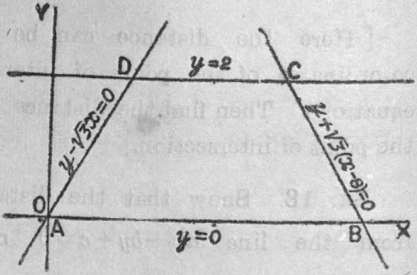


Fig. 26

$\therefore$ Two sides of the quadrilateral formed by the given st. lines are parallel and two are oblique. $\therefore$ It is a trapezium.

Again, since the oblique sides (2) and (4) make equal angles with the parallel sides, the trapezium is cyclic.

Let (1) and (4) intersect at A, (1) and (2) at B, (2) and (3) at C, and (3) and (4) at D.

Solving (1) and (4) the co-ordinates of A are found to be $(0,0)$;

Similary solving (1) and (2), B's co-ordinates are $(8, 0)$,

Solving (3) and (2), C's co-ordinates are $\left(8-\frac{2}{\surd 3}, 2\right)$ and solving (2) and (4), D's co-ordinates are $\left(\frac{2}{\surd 3}, 2\right)$.

$$\therefore \quad AB=8, \text{ and } CD=\sqrt{\left(8-\frac{2}{\surd 3}-\frac{2}{\surd 3}\right)^2+(2-2)^2}$$

$$=\sqrt{\left(8-\frac{4}{\surd 3}\right)^2}=8-\frac{4}{\surd 3}.$$

$\therefore$ the area of the trapezium $=\frac{1}{2}\times$ sum of the parallel sides $\times$ the distance between the parallel sides

$$=\tfrac{1}{2}\left(8+8-\frac{4}{\surd 3}\right)\times 2=16-\frac{4}{\surd 3}=\frac{4}{3}(12-\surd 3) \text{ sq. units.}$$

✓ **Ex. 20.** Calculate the area of the triangle of which two vertices are $(0, 0)$ and $(9, 0)$ and the third vertex is the point of intersection of the lines $x+y-8=0$ and $7x-2y-2=0$.

Let two vertices be A$(0, 0)$ and B$(9, 0)$. Then AB $=9$ units. If C be the third vertex, its co-ordinates are found to be $(2, 6)$ by solving $x+y-8=0$ and $7x-2y-2=0$.

As AB coincides with the x-axis, the perpendicular distance of C from AB is equal to the ordinate of the point C, *i.e.*, 6 units.

$\therefore$ The area of the $\triangle ABC=\frac{1}{2}\times 9\times 6$ sq. units

$=27$ square units.

Exercise 4

1. (a) Find the equation to the line parallel to the x-axis and passing through the point $(4, 7)$.

(b) Find the equation to the line passing through the point $(-3, 4)$ and parallel to the y-axis.

2. Find the angle between the lines :

(i) $2y - x = 3$ and $y = \frac{1}{3}x + 5$.

(ii) $y = 2x + 3$ and $3y = x + 6$.

(iii) $ax - by + c = 0$ and $(a-b)x - (a+b)y + c = 0$.

(iv) $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$.

(v) $x \cos 25^\circ + y \sin 25^\circ - 7 = 0$ and $x \sin 25^\circ - y \cos 25^\circ + 7 = 0$.

3. Find the co-ordinates of the point of intersection of the lines :

(i) $2x - 3y + 5 = 0$ and $7x + 4y - 3 = 0$. [Utkal, 1947]

(ii) $\frac{x}{3} + \frac{y}{2} = 1$ and $\frac{x}{2} + \frac{y}{3} = 1$. [C. U. 1943]

(iii) $\frac{x}{4} + \frac{y}{5} = 1$ and $\frac{x}{5} + \frac{y}{4} = 1$. [C. U. 1941]

Also find the equation to the straight line through this point of intersection and cutting both the axes at an angle of 45°.

(iv) $x \cos \theta + y \sin \theta = p$ and $x \cos \phi + y \sin \phi = p$.

4. Find the equation to the straight line passing through—

(a) 'the point $(3, 5)$ and parallel to the line $4x - 3y + 1 = 0$. [C. U. 1947]

(b) the point $(3, 4)$ and perpendicular to the line $4x - 3y + 1 = 0$. [C. U. 1956]

(c) the point (3, 2) and the point of intersection of the lines $3x+y-5=0$ and $x+5y+3=0$. [C. U. 1942]

(d) the point (1, 2) and the point of intersection of the lines $x+3y+1=0$ and $2x+7y+3=0$. [C. U. 1946]

(e) the origin and the intersection of the straight lines $2x+3y=1$ and $x-y=2$. [C. U. 1933]

(f) the point of intersection of $25x+41y-8=0$ and $5x+7y+9=0$, and parallel to the line $2x+3y+7=0$. [U. P. B. 1941]

(g) the point of intersection of $x+2y=0$ and $y+4x+7=0$ which is perpendicular to the straight line $3x-y=0$. [C. U. 1932]

(h) the point of intersection of the lines $x+2y+3=0$ and $3x+4y+7=0$ and is perpendicular to the straight line $y-x=8$. [C. U. 1940 ; U. P. B. 1949]

(i) the origin and the point of intersection of the lines $\frac{x}{3}+\frac{y}{4}=1$ and $\frac{x}{4}+\frac{y}{3}=1$. [U. P. B. 1948]

(j) the point of intersection of the lines $2x-3y+4=0$ and $3x+4y-5=0$, and perpendicular to the straight line $6x-7y+8=0$. [C. U. 1930 and '44]

(k) the point of intersection of the lines $2x-y+5=0$ and $x+y+1=0$ and the point of intersection of $2x+y-5=0$ and $x-y-7=0$.

5. Prove that the following sets of three lines are concurrent ; also find the respective points of concurrence.

(a) $x+y+1=0$; $2x+3y+1=0$; $3x+4y+2=0$.

(b) $4x-3y-31=0$; $7x-5y-56=0$; $11x-9y-80=0$.

(c) $\frac{x}{a}+\frac{y}{b}=1$; $\frac{x}{b}+\frac{y}{a}=1$; $x=y$.

(d) $ax+(b+c)y-d=0$; $bx+(c+a)y-d=0$:
$cx+(a+b)y-d=0$.

(e) $2x-7y+11=0$; $3x-2y+1=0$; $x-12y+21=0$.
[C. U. 1945]

6. (a) Find the value of p so that $3x+y-2=0$, $px+2y+3=0$ and $2x-y-3=0$ may be concurrent.

(b) Find the value of k for which the three lines $2x-3y+k=0$, $3x-4y-1=0$ and $4x-5y-2=0$ may be concurrent.

(c) Show that the lines joining the origin to the point (2, 3) is concurrent with the straight lines $5x-3y=2$ and $x+y=10$.
[Andhra. 1947]

(d) Verify that the three lines $x-y-7=0$, $x+2y+6=0$ and $2x+y-1=0$ pass through a common point and that this point is equidistant from (5, −4), (3, −2) and (1, −6).
[C. U. 1956]

7. Prove that the three lines given by $ax+by+c=0$, $bx+cy+a=0$ and $cx+ay+b=0$ will be concurrent if $a+b+c=0$.

8. Prove that the following pairs of lines are parallel

(a) $3x+2y+5=0$ and $6x+4y-7=0$.

(b) $\frac{x}{a}+\frac{y}{b}=1$ and $bx+ay=c$.

9. Prove that the following pairs of lines are perpendicular to each other :

(a) $4x-5y+7=0$ and $10x+8y+3=0$.

(b) $\frac{x}{4}+\frac{y}{5}=1$ and $\frac{2x}{5}-\frac{y}{2}=1$.

10. Find the equation of the straight line passing through the point (3, 2) and the intersection of the lines $3x+y-5=0$

and $x+5y+3=0$. Find also the area of the triangle cut off from the co-ordinate axes by this line. [C. U. 1942]

11. Find the equation of the line which divides internally the line joining $(-3, 7)$ to $(5, -4)$ in the ratio 4 : 7 and is perpendicular to this line.

12. Show that the area of the triangle formed by the straight lines whose equations are $y=m_1x+c_1$, $y=m_2x+c_2$ and $x=0$ is $\frac{1}{2}\,\frac{(c_1-c_2)^2}{m_2-m_1}$. [C. U. 1955]

13. Show that the lines $(a+b)x+(a-b)y-2ab=0$, $(a-b)x+(a+b)y-2ab=0$ and $x+y=0$ form an isosceles triangle whose vertical angle is $2\tan^{-1}\left(\frac{a}{b}\right)$.

Determine the co-ordinates of its centroid. [C. U.]

14. (*a*) Prove that the diagonals of the parallelogram formed by the four straight lines $\sqrt{3}x+y=0$, $\sqrt{3}y+x=0$, $\sqrt{3}x+y=1$ and $\sqrt{3}y+x=1$ are at right angles to one another. [C. U. 1953]

(*b*) Prove that the diagonals of the parallelogram formed by the four straight lines $\frac{x}{a}+\frac{y}{b}=1$, $\frac{x}{b}+\frac{y}{a}=1$, $\frac{x}{a}+\frac{y}{b}=2$ and $\frac{x}{b}+\frac{y}{a}=2$ are at right angles to one another. [C. U.]

15. Find the equations to the straight lines—

(*a*) which pass through $(3, 2)$ and are inclined at an angle of 45° to the straight line $x=2y+4$.

(*b*) passing through $(7, 9)$ and inclined at an angle of 60° to the straight line $x-\sqrt{3}y-2\sqrt{3}=0$.

(*c*) which pass through the origin and are inclined at 75° to the straight line $x+y+\sqrt{3}(y-x)=a$.

(*d*) Find the equation to the line through the origin perpendicular to $x\cos\theta+y\sin\theta=p$.

16. Through the point (3, 4) are drawn two straight lines each inclined at 45° to the straight line $x-y=2$. Find their equations and find the area included by the three lines.

17. Find the equations to the two straight lines which pass through the point (4, 5) and make equal angles with the two straight lines $3x=4y+7$ and $5y=12x+6$.

$\frac{m-\frac{3}{4}}{1+\frac{3m}{4}} = \frac{\frac{12}{5}-m}{1+\frac{12m}{5}}$

or $63m^2-32m-63=0$

or $63m^2-81m+49m-63=0$

$\therefore m=\frac{9}{7}$

or $m=-\frac{7}{9}$

(18) Two straight lines pass through the point $(-2, 5)$ such that one of them makes an angle of $\tan^{-1}\frac{3}{4}$ with the given line $x-y+5=0$ and the given line makes an angle of $\tan^{-1}\frac{2}{3}$ with the other line. Find the equations to the two lines.

19. (*a*) Find the perpendicular distance from the origin of the perpendicular from the point (1, 2) upon the straight line $x-\sqrt{3}y+4=0$.

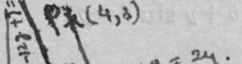
a. (*b*) Find the equations of two straight lines each 5 inches distant from the origin and inclined at an angle of 30° to the x-axis. What lengths do these lines intercept on the axes? [J. B. A.]

20. (*a*) Find the distance from (3, 8) measured along the line $4x-3y+12=0$ to the point where this line intersects the line $4x+5y=60$.

(*b*) Find the distance from the point $(-2, 7)$ measured along the straight line $\frac{x}{12}+\frac{y}{6}=1$ upto its point of intersection with the line $y=x+2$.

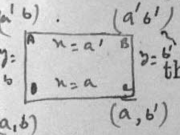
21. Find the equation to the straight line which passes through the point P(4, 3) and is parallel to the line

a. $5x-12y+7=0$; also determine the length intercepted on this line between the point P and the straight line $x+y=24$.



22. Find the area of the triangle, two of whose vertices are the points $(3, 0)$ and $(15, 0)$ and the third vertex is the point of intersection of the lines $4x+3y=0$ and $5x+4y-1=0$.

23. Show that the lines $y=0$, $\sqrt{3}y+x-10=0$, $y=\sqrt{3}$ and $\sqrt{3}y-x=0$ form a cyclic trapezium. Calculate the co-ordinates of the vertices and also the area of the trapezium.



24. Find the equations to the diagonals of the rectangle the equations of whose sides are $x=a$, $x=a'$, $y=b$ and $y=b'$. [C. U. 1951]

25. Show that the point of intersection of the straight lines $\frac{x}{a}+\frac{y}{b}=1, \frac{x}{b}+\frac{y}{a}=1$ is the vertex of a square whose adjacent sides are along the axes of co-ordinates unless $a+b=0$. [C. U. 1952]

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Straight Lines

29. To find the position of a point in relation to a given straight line.

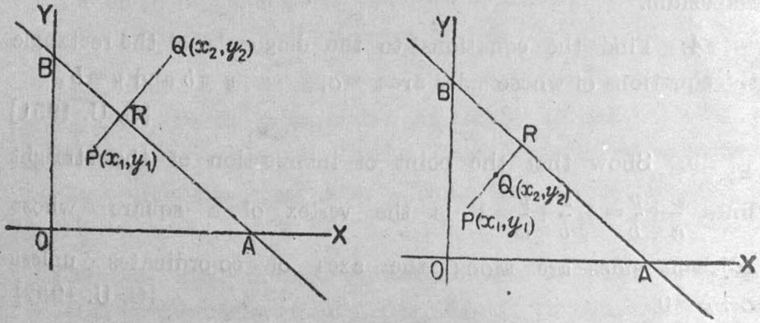


Fig. 27

Let the equation of the given st. line be $ax+by+c=0$, and let it cut the axes at A and B respectively.

(i) Suppose that the points P(x_1, y_1) and Q(x_2, y_2) lie on the opposite sides of AB [first fig.].

Join PQ, cutting AB at R. Suppose PQ is divided internally at R in the ratio $m : n$, *i.e.*, $\frac{\text{PR}}{\text{QR}}=\frac{m}{n}$.

$\therefore$ The co-ordinates of R are $\left(\frac{mx_2+nx_1}{m+n},\ \frac{my_2+ny_1}{m+n}\right)$.

$\because$ R is on AB, $\therefore$ its co-ordinates will satisfy

$$ax+by+c=0, \quad \therefore \quad a.\frac{mx_2+nx_1}{m+n}+b.\frac{my_2+ny_1}{m+n}+c=0,$$

$\therefore$ On simplification we have $\frac{ax_1+by_1+c}{ax_2+by_2+c}=-\frac{m}{n} \cdots (1)$

Again, suppose that P(x_1, y_1) and Q(x_2, y_2) are on the same side of AB [second fig.].

Join PQ. Here PQ produced cuts AB at R, so PQ is divided externally at R in the ratio $m : n$ (say), *i.e.*, $\frac{\text{PR}}{\text{QR}}=\frac{m}{n}$.

$$\therefore \text{ the co-ordinates of R are } \left(\frac{mx_2-nx_1}{m-n},\ \frac{my_2-ny_1}{m-n}\right).$$

$$\because \text{ R lies on AB}, \quad \therefore\ a.\frac{mx_2-nx_1}{m-n}+b.\,\frac{my_2-ny_1}{m-n}+c=0,$$

$$\text{or,}\quad \frac{ax_1+by_1+c}{ax_2+by_2+c}=+\frac{m}{n}\ [\text{ simplifying }]\ldots(2).$$

$\because$ $\frac{m}{n}$ is positive, $\therefore$ the ratio $\frac{ax_1+by_1+c}{ax_2+by_2+c}$ is negative in (1) and positive in (2). Hence it is evident that in the first case the expressions ax_1+by_1+c and ax_2+by_2+c are of opposite signs and in the second case both of them are of the same sign.

Hence, the points (x_1, y_1) and (x_2, y_2) will be on the same side of the line $ax+by+c=0$, when ax_1+by_1+c and ax_2+by_2+c are of the same sign, but the points will lie on the opposite sides of the line, when the expressions are of opposite signs.

Corollary : To find on which side of a given st. line a certain point is, we have to determine whether the point and the origin are on the same side of the line or on opposite sides of it.

To determine it, find the signs of the expression of the given equation by putting the co-ordinates of the origin (0, 0) and of the given point in it. If the two signs be the same, then the given point is on the origin side (*i.e.*, on the same side with the origin), but if the signs obtained be opposite to each other, then the point is on the non-origin side (*i.e.*, on the side opposite to the origin).

30. To find the length of the perpendicular from a given point to a given straight line.

Let OM be the perpendicular from the origin O to the given st. line AB and let $\angle XOM=\theta$. Let $P(x_1, y_1)$ be the given point.

Draw PN through P parallel to AB, cutting OM produced at N. Then ON is perpendicular to PN. Hence, the length of the perpendicular from P to AB = MN.

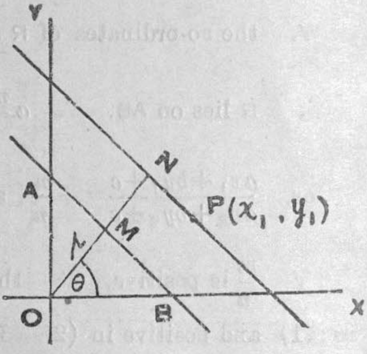


Fig. 28

Suppose $OM=p$ and $ON=p_1$.

(1) The equation of the st. line AB is

$x\cos\theta+y\sin\theta=p$ (where $p=OM$).

Again, $\because$ $ON=p_1$ and PN ∥ AB,

$\therefore$ the equation of PN is $x\cos\theta+y\sin\theta=p_1$.

$\because$ the st. line passes through the pt. (x_1, y_1),

$\therefore$ $x_1\cos\theta+y_1\sin\theta=p_1$.

$\therefore$ the required length of the perpendicular

$$=MN=ON-OM=p_1-p=\boldsymbol{x_1\cos\theta+y_1\sin\theta-p}\cdots(1)$$

(2) Suppose the equation of AB to be $ax+by+c=0$

[the equation being written so that c is a negative quantity.]

$$\therefore\quad \frac{a}{\sqrt{a^2+b^2}}x+\frac{b}{\sqrt{a^2+b^2}}y+\frac{c}{\sqrt{a^2+b^2}}=0,$$

$$\therefore\quad \cos\theta=\frac{a}{\sqrt{a^2+b^2}},\ \sin\theta=\frac{b}{\sqrt{a^2+b^2}},\ -p=\frac{c}{\sqrt{a^2+b^2}}.$$

$\therefore$ the length of the perpendicular from $P(x_1, y_1)$

$$=x_1 \cos \theta + y_1 \sin \theta - p$$

$$=\frac{a}{\sqrt{a^2+b^2}}x_1+\frac{b}{\sqrt{a^2+b^2}}y_1+\frac{c}{\sqrt{a^2+b^2}}=\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}.$$

N. B. (i) Here, if the point P (so, the st. line PN) be between the origin O and AB, then the length of the perpendicular (*i.e.*, MN)$=p-p_1=-(p_1-p)$.

Hence, the length of the perpendicular from p

$$=\pm(p_1-p)=\pm\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}.$$

We notice here that if the origin O is on the same side of AB and the pt. P (so PN), their x- and y- intercepts are of the same sign. In both such cases the length of the perpendicular is equal to the difference of p and p_1.

Both the st. lines AB and PN may cut the axes being in any one of the quadrants.

(ii) If the origin O lie between AB and PN (*i.e.* the pt. P), then their x- and y- intercepts are of opposite signs. Here, the length of the perpendicular is equal to the sum of p_1 and p.

$\therefore$ the length of the perpendicular $=\pm\dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$.

In such cases one of the st. lines AB and PN may lie in the first and the other in the third quadrant or one in the second and the other in the fourth quadrant.

(iii) The co-ordinates of the origin are (0, 0). So we may have the length of the perpendicular from the origin to the st. line $ax+by+c=0$ (*i.e.* AB) by putting $x_1=0$ and $y_1=0$ in

$$\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}.$$

Then the length of the perpendicular $=\dfrac{c}{\sqrt{a^2+b^2}}$.

(iv) As in trigonometry the line describing an angle is taken to be positive, so in co-ordinate geometry the perpendicular (ON) from the origin to a st. line is always taken to be positive by convention.

Hence, when a point is on the same side of a st. line as the origin, the perpendiculars from the point and the origin have the same sign. So the perpendicular from the point is positive.

When the point and the origin are on opposite sides of the line, the perpendiculars are of opposite signs. So the perpendicular from the point is negative.

In applying this method the given equation should be written so that the constant term (*i.e.*, the term independent of x and y) is positive.

31. To find the equation of the straight line bisecting the angle between two straight lines.

Let XOX′ and YOY′ be the axes and O the origin. Let the st. lines AB and CD intersect at K and let their equations be $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ respectively.

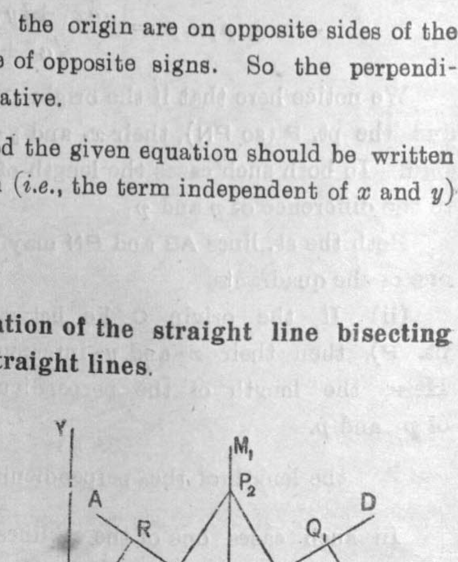


Fig. 29

Let MN and M_1N_1 be respectively the bisectors of the acute angle and the obtuse angle between AB and CD.

Suppose P to be a point on the bisector of the angle between AB and CD and let its co-ordinates be (X, Y). Then the perpendicular distances of P from AB and CD are equal.

$$\therefore \quad \frac{a_1X+b_1Y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2X+b_2Y+c_2}{\sqrt{a_2^2+b_2^2}}\cdots(1)$$

Now, if P is on the bisector MN (say at the pt. P_1), then the signs of the perpendiculars from P_1 to AB and CD are both like (same) or both unlike (opposite) the signs of the perpendiculars from O (*i.e.*, both will be either positive or negative).

Again, if P is on the bisector M_1N_1 (say at P_2), then only one perpendicular from P_2 will be like the perpendicular from O, while the other will be unlike, *i.e.*, one of them will be positive and the other negative.

Now, $\dfrac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\dfrac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$, being equations of the first degree in x and y, represent two st. lines. As these equations are satisfied by the co-ordinates (X, Y), the point (X, Y) lies on these st. lines.

Hence, these are the equations of the bisectors of the angle between the given lines.

[**N. B.** In the given figure P_1R_1 and P_1Q_1 being either both positive or both negative, the equation of MN is

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}.$$

Again, of P_2R_2 and P_2Q_2, one is positive and the other is negative, so the equation of M_1N_1 is

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}.\ ]$$

Examples (5)

Ex. 1. Find on which side of the straight line $x-2y-5=0$ lies the point P$(-3, -5)$.

Writing the independent term (the constant term) of the given equation as a positive quantity we have $-x+2y+5=0$.

Putting the co-ordinates of P (*i.e.*, $x=-3$, $y=-5$) in the lefthand member of the equation we have

$-(-3)+2(-5)+5=3-10+5=-2$ (a negative quantity),

$\therefore$ the point is on the non-origin side, *i.e.*, on the side of the st. line opposite to the origin.

Ex. 2. Find whether the points A$(1, 1)$ and B$(5, -3)$ lie on the same side or on opposite sides of the line $3x+4y-5=0$.

Putting the co-ordinates of A $(1, 1)$ in the expression $3x+4y-5$, we have $3x+4y-5=3\times1+4\times1-5=7-5=+2$, which is positive.

Again, putting the co-ordinates of B$(5, -3)$ in it, we have $3\times5+4\times(-3)-5=15-12-5=-2$, which is negative.

Hence A and B lie on the opposite sides of the given line.

Ex. 3. Find the length of the perpendicular from the point $(3, 1)$ on the line $5x-12y+1=0$.

By the formula the length p of the perp. $=\dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$,

Here $a=5$, $b=-12$, $x_1=3$, $y_1=1$.

$\therefore$ the required length $=\dfrac{5\times3-12\times1+1}{\sqrt{5^2+(-12)^2}}=\dfrac{4}{\sqrt{13^2}}=\dfrac{4}{13}$.

[**N. B.** In finding the length of the perpendicular, it may sometimes be negative and the negative value should be given as the required length (vide Art. 30 (IV)].

Ex. 4. Find the equations of the bisectors of the angles between the lines $12x-5y+4=0$ and $3x-4y-2=0$.

Writing the constant terms (4 and -2) of the two equations as positive quantities, we have $12x-5y+4=0$ and $-3x+4y+2=0$.

The equation of the bisector of the angle in which the origin lies is $\dfrac{12x-5y+4}{\sqrt{(12)^2+(-5)^2}}=+\dfrac{-3x+4y+2}{\sqrt{(-3)^2+(4)^2}}$,

or, $\dfrac{12x-5y+4}{13}=\dfrac{-3x+4y+2}{5}$,

or, $60x-25y+20=-39x+52y+26$, or, $99x-77y-6=0$.

Again, the equation of the bisector of the other included angle is $\dfrac{12x-5y+4}{\sqrt{12^2+(-5)^2}}=-\dfrac{-3x+4y+2}{\sqrt{(-3)^2+4^2}}$,

or, $60x-25y+20=39x-52y-26$, or, $21x+27y+46=0$.

✓**Ex. 5.** Find the foot of the perpendicular from the point $(2, -2)$ to the line $3x-y+2=0$.

Here, the given equation is $3x-y+2=0$, or, $y=3x+2$.

$\therefore$ its gradient $m_1=3$.

$\therefore$ the gradient (m_2) of any st. line perpendicular to the line $y=3x+2$ must be $-\frac{1}{3}$ [$\because m_1m_2=-1$]

$\therefore$ the equation of the perpendicular from the point $(2, -2)$ to the line $y=3x+2$ is $y-(-2)=-\frac{1}{3}(x-2)$,

or, $x+3y+4=0$.

Now, the foot of the perpendicular is evidently the point of intersection of the lines $y=3x+2$ and $x+3y+4=0$.

Solving the two equations we have $x=-1$, $y=-1$.

$\therefore$ the foot of the perpendicular is the point $(-1, -1)$.

✓**Ex. 6.** Find the distance between the parallel lines $4x+3y=8$ and $4x+3y+12=0$.

$\because$ the given lines are parallel, $\therefore$ the perpendicular drawn from the origin to one of them is also perpendicular to the other.

The length of the perpendicular from the origin$(0,0)$to the line $4x+3y=8$, or, $4x+3y-8=0$ is $\dfrac{-8}{\sqrt{4^2+3^2}}$ or, $-\dfrac{8}{5}$.

The length of the perpendicular from $(0, 0)$ to the other line

$4x+3y+12=0$ is $\dfrac{12}{\sqrt{4^2+3^2}}$ or $\dfrac{12}{5}$.

$\because$ the lengths of the two perpendiculars are of opposite signs,

$\therefore$ the given st. lines are on the opposite sides of the origin.

$\therefore$ the distance between the st. lines $=\dfrac{12}{5}+\dfrac{8}{5}=4$.

[**Otherwise**] The distance between two st. lines is the same as the length of the perpendicular drawn from any point on one of them to the other st. line.

Putting $y=0$ in the first equation, we have $x=2$.

$\therefore$ the point $(2, 0)$ is a point on this st. line.

Now, the perpendicular distance of the line $4x+3y+12=0$ from the point $(2, 0)=\dfrac{4x+3y+12}{\sqrt{4^2+3^2}}=\dfrac{4\times2+3\times0+12}{5}=\dfrac{20}{5}=4$.

$\therefore$ the distance between the lines $=4$.

Ex. 7. Find the equation to the straight line passing through the origin and the point of intersection of the lines $x-y=4$ and $y+7x+20=0$, and prove that it bisects the angle between them. [U. P. B. 1951]

The equation of any st. line passing through the point of intersection of the lines $x-y=4\cdots(1)$ and $y+7x+20=0\cdots(2)$ is $x-y-4+k(y+7x+20)=0$ [k being an arbitrary constant],

or, $(7k+1)x+(k-1)y+20k-4=0\cdots(3)$

$\because$ the line-(3) passes through the origin,

$\therefore$ its constant term must be zero.

$\therefore$ We have $20k-4=0$, $\therefore$ $k=\frac{4}{20}=\frac{1}{5}$.

Hence the required equation is $x-y-4+\frac{1}{5}(y+7x+20)=0$,

or, $y=3x\ldots(4)$.

$\because$ The st. line-(4) passes through the origin, $\therefore$ to bisect the angle between (1) and (2), it must be the bisector of that angle between them in which the origin lies.

Arranging the given equations so that their independent terms are positive, we have $y-x+4=0$ and $y+7x+20=0$.

$\therefore$ The equation of the bisector of the angle between them in which the origin lies is $\frac{y-x+4}{\sqrt{1^2+(-1)^2}}=+\frac{y+7x+20}{\sqrt{1^2+7^2}}$,

or, $\frac{y-x+4}{\sqrt{2}}=\frac{y+7x+20}{5\sqrt{2}}$, or, $\frac{y-x+4}{1}=\frac{y+7x+20}{5}$,

or, $5y-5x+20=y+7x+20$, or, $12x=4y$, or, $y=3x$.

Hence, the st. line, passing through the origin and the point of intersection of the given lines, bisects the angle between the lines.

Ex. 8. If the three lines $x+y=0$, $x-3y=0$ and $x-2y=1$ form a triangle, find the equation of the perpendicular let fall on $x+y=0$ from the opposite vertex.

Here the vertex of the triangle opposite to the side $x+y=0\ldots(1)$ is the point of intersection of the other two equations.

The equation of any line through the point of intersection of the lines $x-3y=0$ and $x-2y-1=0$ is
$x-3y+k(x-2y-1)=0$, or, $(k+1)x-(2k+3)y-k=0\ldots(2)$.
Now, the lines (1) and (2) will be perpendicular to each other, if $(k+1)\times 1-(2k+3)\times 1=0$ [Vide Art. 24 (B)], *i.e.*, if $k=-2$.

Hence, the required equation is $x-3y-2(x-2y-1)=0$,

or, $y-x+2=0$.

Ex. 9. Find the orthocentre of the triangle whose vertices are $(1, 5)$, $(7, 2)$ and $(4, 9)$.

Let the co-ordinates of the vertices of $\triangle ABC$ be $A(1, 5)$, $B(7, 2)$ and $C(4, 9)$.

The equation of any line through $A(1, 5)$ will be $y-5=m(x-1)\cdots(1)$.

The gradient of $BC=\dfrac{9-2}{4-7}=-\dfrac{7}{3}$.

$\therefore$ the st. line-(1) will be perpendicular on BC, when $m=\dfrac{3}{7}$.

$\therefore$ The equation of the perpendicular on BC passing through the vertex A is $y-5=\frac{3}{7}(x-1)$, or, $3x-7y+32=0\cdots(2)$.

Again, the equation of any line passing through $B(7, 2)$ is $y-2=m(x-7)$.

The gradient of the side $AC=\dfrac{9-5}{4-1}=\dfrac{4}{3}$.

$\therefore$ The equation of the perpendicular on AC passing through the vertex B is $y-2=-\frac{3}{4}(x-7)$,

or, $3x+4y-29=0\cdots(3)$.

Now, the point of intersection of (2) and (3) will be the orthocentre of $\triangle ABC$.

Solving (2) and (3) we have $x=\frac{25}{11}$ and $y=\frac{61}{11}$.

$\therefore$ the co-ordinates of the orthocentre are $(\frac{25}{11}, \frac{61}{11})$.

✓**Ex. 10.** Prove that in any triangle perpendiculars drawn from the vertices upon the opposite sides are concurrent.

Suppose the co-ordinates of the vertices A, B, C of the $\triangle ABC$ are (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively.

The gradient of the side BC $= \dfrac{y_3 - y_2}{x_3 - x_2}$.

$\therefore$ The equation of the perpendicular from the vertex A to the side BC is $y - y_1 = -\dfrac{x_3 - x_2}{y_3 - y_2}(x - x_1)$,

or, $y(y_3 - y_2) + x(x_3 - x_2) - \{y_1(y_3 - y_2) + x_1(x_3 - x_2)\} = 0 \ldots (1)$.

Similarly the equations of the perpendiculars from B and C to CA and AB are respectively

$$y(y_1 - y_3) + x(x_1 - x_3) - \{y_2(y_1 - y_3) + x_2(x_1 - x_3)\} = 0 \cdots (2)$$

and $y(y_2 - y_1) + x(x_2 - x_1) - \{y_3(y_2 - y_1) + x_3(x_2 - x_1)\} = 0 \ldots (3)$

Now, we find that the sum of the left hand sides of (1), (2) and (3) becomes 0.

$\therefore$ the lines (1), (2) and (3) are concurrent [Vide Art. 28(B), here $p = q = r = 1$]

Hence, the perpendiculars from the vertices of the triangle to the opposite sides are concurrent.

✓**Ex. 11.** Find the internal bisectors of the angles of the triangle whose sides are $x = 0$, $y = 0$ and $3x + 4y - 12 = 0$. [C. U.]

Here $x = 0 \ldots (1)$ is the equation of the y-axis, and $y = 0 \cdots (2)$ is the equation of the x-axis and the st. line $3x + 4y - 12 = 0 \ldots (3)$ cuts x-axis and y-axis on their positive sides.

Evidently the intercepts on the two axes are two sides of the triangle.

∴ The bisector of the right angle between the axes in the first quadrant is the bisector of one angle of the triangle. As this bisector passes through the origin, its constant term must be zero and it must make an angle of 45° with the positive direction of the x-axis.

∴ its gradient $m = \tan 45^\circ = 1$.

∴ The equation of that bisector is $y = x$.

Again, the equation of the bisector of the angle between the lines (1) and (2) is $x = \dfrac{-3x-4y+12}{\sqrt{3^2+4^2}}$, or, $5x = -3x-4y+12$,

$$\text{or,}\quad 2x+y-3=0.$$

The equation of the bisector of the angle between the lines (2) and (3) is $y = \dfrac{-3x-4y+12}{\sqrt{3^2+4^2}}$, or, $5y = -3x-4y+12$,

$$\text{or,}\quad x+3y-4=0.$$

Hence, the equations of the internal bisectors of the angles of the triangle are $y = x$, $2x+y-3=0$ and $x+3y-4=0$.

Exercise 5

1. (*a*) Find on which side of the straight line $3x+4y+5=0$ lies the point $(2, -2)$.

(*b*) Find on which side of the straight line $5x-9y+7=0$ lies the point $(0, 3)$.

2. (*a*) Find whether the points A $(0, -4)$ and B $(-3, 1)$ lie on the same side or on the opposite sides of the line $6x+7y+12=0$.

(*b*) Find whether the points P $(3, 1)$ and Q $(-4, -1)$ lie on the same side or on the opposite sides of the line $3x-4y+7=0$.

3. (*a*) Find the distance of the point $(-3, 4)$ from the line $2x-3y+1=0$.

(*b*) Find the distance of the point $(1, 0)$ from the straight line $5x+12y-8=0$.

4. Find the lengths of the altitudes of the triangle having the vertices $(-2, 1)$, $(1, 4)$ and $(3, -1)$.

5. Show that the point $(1, 1)$ is equidistant from the lines $3x+4y=12$, $5x-12y+20=0$ and $4x-3y=6$.

✓6. If the sum of the perpendiculars dropped from a variable point P on the two str. lines $x+y-5=0$ and $3x-2y+7=0$ be always equal to 10, prove that P must move on a right line.
[C. U. 1950].

7. In $\triangle ABC$, $2x+y+1=0$, $2x+3y+1=0$ and $3x+4y+3=0$ represent the sides BC, CA and AB respectively. Find the equation of the altitude through A.

8. Find the orthocentre of the triangle whose sides are given by $x-y+1=0$, $3x+y-17=0$, $x+5y+13=0$.

9. Find the foot of the perpendicular from the point $(3, -2)$ to the straight line $2x-y+7=0$.

10. Find the equations to the straight lines bisecting the angles between the following pairs of straight lines :—

✓(*a*) $8x-6y+11=0$ and $12x-5y-6=0$

(*b*) $4y+3x-12=0$ and $3y+4x-24=0$

(*c*) $x\cos\theta+y\sin\theta=p_1$ and $x\cos\phi+y\sin\phi=p_2$.

11. Find the distance between the parallel lines

✓(*a*) $3x+4y=6$ and $3x+4y+5=0$.

(*b*) $y=mx+c_1$ and $y=mx+c_2$.

✓ **12.** What are the points on the axis of x whose perpendicular distance from the straight line $\frac{x}{a}+\frac{y}{b}=1$ is a ?

[C. U. 1951]

✓ **13.** Find the equations of the two str. lines drawn through the point $(0, a)$, on which the perpendiculars let fall from the point $(2a, 2a)$ are each of length a.

[C. U. 1953]

✓ **14.** Express the condition that the perpendicular dropped from the point $(3, -2)$ on the line $lx+my+n=0$ may be of constant length 5. [C. U. 1956]

✓ **15.** Show that the perpendiculars let fall from any point of the str. line $7x-9y+10=0$ upon the two str. lines $3x+4y=5$ and $12x+5y=7$ are equal to each other.

[C. U. 1952]

✓ **16.** Find the equation of the str. line which lies midway between the point $(2, -1)$ and the str. line $3x-2y+5=0$.

[J. B. A.]

ANSWERS

MENSURATION

Exercise 1

1. 52 sq. ft., 24 cu. ft., $\sqrt{29}$ ft. **2.** 150 sq. ft., 125 cu. ft.,
$5\sqrt{3}$ ft. **3.** 54 sq. ft., 27 cu. ft. **4.** 2·5 ft. **5.** 236 sq. ft.
6. 5196·15 cu. in. (app.) **7.** 15 in., 9 in., 6 in.
8. 13 in. **9.** 11 ft. **10.** 12288 **11.** Rs. 6.13 *a.* 9 *p.*
12. 9 cu. ft. ; 3182 cu. in. **13.** 25 ft. **14.** 10·392 in.
15. $\frac{16}{21}$ oz, $1\frac{7}{9}$ oz. **17.** $314\frac{1}{2}$ cu. ft. **18.** 91 cu. yd. 7 cu. ft.

Exercise 2

1. 3 sq. ft. **2.** 432 cu. cm. **4.** $13\frac{1}{3}$ ft., 172 sq. ft.
5. 10 ft. **6.** 935·3 cu. ft. **7.** 880 cu. ft.
8. 370·764 sq. ft. **9.** 6363·96 cu. ft. **10.** 1 sq. ft.
11. 600 cu. ft. **12.** $180\sqrt{3}$ cu. ft. **14.** 240 cu. cm.
15. 80 sq. ft., 64 cu. ft. **16.** 360 cu. cm., 432 sq. cm.
17. Rs. 5. **18.** 4 cm., 144 cu. cm. **19.** 8 cm., 1152 cu. cm.

Exercise 3

1. (1) 13 in. (2) $204\frac{2}{7}$ sq. in. (3) $282\frac{6}{7}$ sq. in. (4) $314\frac{2}{7}$ cu. in.
2. 30 in., 29·79 in. (App.) **3.** $204\frac{2}{7}$ sq. cm., $314\frac{2}{7}$ cu. cm.
4. 7 ft. **5.** 1 ft. **6.** $427\frac{3}{7}$ sq. cm., $1005\frac{5}{7}$ c. c.
7. 236·28 sq. ft. (App.) **8.** $37\frac{5}{7}$ cu. in. **9.** 1930·971 cu. in.
10. $116\frac{27}{154}$ cu. ft. **11.** $427\frac{3}{7}$ sq. ft., $1005\frac{5}{7}$ cu. ft.
12. $37\frac{5}{7}$ cu. ft., $47\frac{1}{7}$ sq. ft.

ALGEBRA

Exercise 1

1. $a-3b=0$ **2.** $mn=10$ **3.** $ab'-a'b=0$
4. $a^6-a^3=1$ **5.** $(ab+d)^2=(a^2-b+c)(b^2-bc+ad)$
6. $ab=1$ **7.** $c^2-bc+ab^2=0$ **8.** $a^{\frac{2}{3}}-b^{\frac{2}{3}}=4.$ **9.** $ab=6$
10. $a^2+2c-b=0$ **11.** $(a+b)^{\frac{2}{3}}+(a-b)^{\frac{2}{3}}=2$
12. $p^3-3pq+2r=0$ **13.** $m^3-3lm-n=0$
14. $a^4-2a^2b^2-b^4+2c^4=0$
15. $a(b_1c_2-b_2c_1)+b(c_1a_2-c_2a_1)+c(a_1b_2-a_2b_1)=0$
16. $a^2+b^2+c^2+2abc=1$ **17.** $b^4c^4+c^4a^4+a^4b^4=a^2b^2c^2d^2$
18. $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=2$ **19.** $x^2+y^2=a^2$
21. $\frac{x^2y^2}{x^2+y^2}=a^2$ **22.** $x^2+y^2+z^2+2xyz=1.$

Exercise 2

1. 34 **2.** 1 **3.** 3 **4.** $a, a+d, a+2d, \cdots$
5. $5, 2, -1, -4, \cdots$ **9.** -26 **7.** $35, 3r-1$
8. $-30, 10-2n$ **9.** $3\frac{1}{4}$ **10.** (i) $a-\frac{1}{n}+1$, (ii) $\frac{n^2-n+1}{n}$
11. 9th **12.** 6th, 11th **13.** No **14.** 16th
15. 38 **16.** -49 **17.** 9, 0 **18.** $2, 5, 8, \cdots$
19. $\frac{d(p-1)-c(q-1)}{p-q}$, $\frac{c-d}{p-q}$ **20.** 0 **21.** $\frac{r(a-b)-(aq-bp)}{p-q}$.

Exercise 3

1. 15 **2.** 4 **3.** -9 **4.** x^2+a^2 **5.** $2\frac{7}{8}$
6. $\frac{m^2+n^2}{m^2-n^2}$ **7.** 6, 11, 16, 21, 26, 31, 36
8. 68, 132, 196, 260 **9.** 6 **10.** 7.

Exercise 4

1. 100 **2.** $\frac{n}{2}(3n+7)$ **3.** -96 **4.** 290 **5.** $61\frac{3}{4}$

6. $\frac{n+1}{2}$ **7.** 378 **8.** $(n-1)(2n-3)$ **9.** $\frac{1}{2}(n-1)(n+2)$

10. $19(\sqrt{2}+18)$ **11.** 3927 **12.** -144 **13.** 900
14. 19096 **15.** 4080 **16.** $n(2n+1)$ **17.** 247
18. $\frac{1}{2}n(n+1)$ **19.** 6867 **20.** 7500 **21.** 2940
22. 861 **23.** -368 **24.** 1763 **25.** 11
26. 19 **27.** 551 **28.** 1, 2 **29.** 5, 9, 13,$\cdots$
30. 3, 10 ; or, -1, 12 **31.** $(2r-1)^2$ **32.** 2747
35. 5, 12, 19 ; or, 19, 12, 5 **36.** 1, 4, 7, 10 or, 10, 7, 4, 1

37. $\frac{n}{2}(a+c)$ **39.** n^2 **40.** $\frac{1}{3}n(4n^2+6n-1)$

41. $\frac{n}{2}(4n^2+17n+21)$ **42.** $\frac{n}{2}(6n^2+21n+23)$

43. $\frac{n}{12}(n+1)(n+2)(3n+5)$ **44.** $\frac{n}{3}(n^2+6n+11)$
45. $\frac{1}{6}n(n+1)(n+2)$ **46.** $\frac{1}{6}n(n+1)(2n+1)$
47. $\frac{1}{4}n(n+5)(n^2+5n+10)$ **48.** $\frac{1}{6}n(n+1)(2n+1)$

49. $\frac{n}{4(n+1)}$ **50.** $\frac{n}{3(n+3)}$ **51** (*a*). $\frac{1}{3}(4n^3+18n^2-n)$

51 (*b*). $\frac{-r(r+1)}{2}$ (if r is even), $\frac{r(r+1)}{2}$ (if r is odd)

55. $2\frac{1}{2}$, $4\frac{1}{2}$, $6\frac{1}{2}$, $8\frac{1}{2}$; or, $8\frac{1}{2}$, $6\frac{1}{2}$, $4\frac{1}{2}$, $2\frac{1}{2}$ **58.** 25 **59.** 9, 16
60. 50500 yds. **61.** 9 days
62. after 36 yrs., $18\frac{1}{2}$ yds. **63.** 10 months **64.** 15 hrs.

Exercise 5

1. 128 **2.** 6561 **3.** $\frac{1024}{3125}$ **4.** $\frac{1}{3^{n-1}}$ **5.** $\frac{1}{2^{n-5}}$

6. $3(-3)^{p-1}$ **7.** $\frac{1}{81}$ **8.** $\frac{1}{32}$ **9.** 1, 2, 4, 8$\cdots$

10. $\pm\frac{1}{256}$ **11.** $\frac{1}{6561}$ **12.** 3, 2 **13.** 9th

14. 7th **15.** 512, 2^{n-1} **16.** $\left(\dfrac{a^{n-q}}{b^{n-p}}\right)^{\frac{1}{p-q}}$.

Exercise 6

1. ± 25 **2.** $\pm\frac{1}{4}$ **3.** ± 9 **4.** ± 4 **5.** $\pm x^2y^2$; ± 9

6. 12, 36, 108 ; or, -12, 36, -108 **7.** 15, 45

8. 225, 2025, 18225 ; or, -225, 2025, -18225

9. $\frac{1}{3}$, 1, 3 or, $-\frac{1}{3}$, 1, -3 **10.** $\frac{27}{8}, \frac{9}{4}, \frac{3}{2}, 1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \frac{32}{243}$
or, $-\frac{27}{8}, \frac{9}{4}, -\frac{3}{2}, 1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \frac{16}{81}, -\frac{32}{243}$

16. 32, 18 or, 18, 32.

Exercise 7

1. 255 **2.** $255\frac{1}{2}$ **3.** $\frac{1}{2}(3^n-1)$ **4.** $\frac{1}{4}(1-3^{2n})$

5. $3\frac{63}{64}$ **6.** $\frac{4}{3}\left(1-\dfrac{1}{4^n}\right)$ **7.** $\dfrac{19682}{3-\sqrt{3}}$ **8.** $1\frac{511}{512}$

9. $\frac{5}{9}\left(1-\dfrac{1}{10^{n-1}}\right)$ **10.** $\dfrac{1}{6}-\dfrac{2}{3.10^n}$ **11.** -255 **12.** $1\frac{255}{256}$

13. $7\frac{127}{128}$ **14.** $2^{n+1}-2$ **15.** $\frac{3}{2}\left(1-\dfrac{1}{3^n}\right)$

16. 1530 **17.** 1092 **18.** $\frac{5}{2}(2^{25}-1)$

19. 25500 **20.** 1, 4, 16.

Exercise 8

3. $\frac{70}{81}(10^n-1)-\dfrac{7n}{9}$ **4.** $\frac{20}{81}(10^n-1)-\dfrac{2n}{9}$

5. $\dfrac{2n}{9}-\dfrac{2}{81}\left(1-\dfrac{1}{10^n}\right)$ **6.** $n.2^n$ **7.** $2^{n+1}-n-2$

8. $\frac{3}{4}(3^n-1)-\dfrac{n}{2}$ **10.** $2^{n+1}-2+n(n+1)$

11. 3, 6, 12, or, 12, 6, 3 **12.** $\dfrac{3^{n+1}-3-2n}{4.3^{n-1}}$

13. 2, 5, 8, or, 26, 5, -16 **16.** $\dfrac{y^2-xz}{x+z-2y}$

19. 4, 8, 16, or, 16, 8, 4.

Exercise 9

1. $\frac{2}{5}$, $2\frac{1}{2}$, $b=\frac{2}{5}a^3$. **2.** 25, ±15, $y=\frac{900}{x^2}$ **3.** 2.

5. $\mathrm{R}=k+k^1v^2$ **6.** (i) $\sqrt[3]{a}$, (ii) t^2

8. $x=\frac{3}{10}.\frac{y}{z}$; 9 **10.** $b=2a+\frac{3}{a}$; 7 **11.** 10 or 8

12. 16 : 49 **13.** $38\frac{1}{2}$ sq. ft. **14.** ±6 **15.** $\frac{27}{5}$ days

16. $20\frac{1}{4}$ ft. **17.** $x\propto\frac{b^4}{a}$ **19.** 256 ft.

20. 20*m*. per hr. **21.** $d=4$. **22.** 64 : 45 **24.** Rs. 325

25. 80 ft. **26.** (i) P = 22 (ii) W = 68.

Exercise 10

1. 1·9445, $\bar{1}$·0913, $\bar{3}$·5127 **2.** 23·01, ·05671, ·06689

3. 6, 10 **4.** 7th **5.** 1·644, 1·524 ; ·7656 **6.** (i) 1·248 (ii) 15·7 (iii) ·01795 **7.** 1·6532126 **8.** ·30103

9. £ 70 **10.** Rs. 847. 14 a. 8 p. (real ans. is Rs. 847 4 a. 8 p.) **11.** 275 **12.** 125·2 acres

13. Rs. 478 (App.) **14.** 6616 (real ans. 6615)

15. 22 yds. **16.** 4 **19.** 1·196 sq. yd. **22.** (i) $x=1·593$ (ii) $x=·03$ (App.) **23.** 6 **26.** $\log x=\frac{1}{5}(a+3b)$, $\log y=\frac{1}{5}(a-2b)$ **27.** 3 **29.** 1·77 (App.)

30. $x=·40703$, $y=5·6568$ (App.) **31.** 1·206.

Exercise 11

1. (*a*) $\pm(4\sqrt{2}+3)$ (*b*) $\pm(3\sqrt{3}-1)$ (*c*) $\pm\frac{1}{2}(1+\sqrt{3})$

(*d*) $\pm\sqrt[4]{2}(\sqrt{3}-\sqrt{2})$ (*e*) $\pm\sqrt[4]{7}(\sqrt{\frac{7}{2}}+\sqrt{\frac{3}{2}})$ (*f*) $\pm\left(3+\frac{\sqrt{7}}{2}\right)$

2. (*a*) $\pm\frac{1}{\sqrt{2}}(\sqrt{3x-1}+\sqrt{x-2})$

(b) $\pm\frac{1}{\sqrt{2}}(\sqrt{x^4+x^2+1}+\sqrt{x^4-x^2+1})$

(c) $\pm(\sqrt{3}+\sqrt{5}+2\sqrt{2})$ (b) $\pm(\sqrt{2}-\sqrt{3}-\sqrt{6})$

(e) $\pm\{\sqrt{(x-y)}+\sqrt{z}\}$

3. (a) $\sqrt[3]{9}-\sqrt[3]{3}+1$

(b) $9\sqrt{3}-9.\sqrt[3]{2}+3\sqrt{3}.\sqrt[3]{4}-6+2\sqrt{3}.\sqrt[3]{2}-2\sqrt[3]{4}$, (c) $\sqrt[3]{2}-1$

(d) $(\sqrt{x}+\sqrt{y}+\sqrt{z})(\sqrt{z}+\sqrt{x}-\sqrt{y})(\sqrt{z}-\sqrt{x}+\sqrt{y})$

4. (a) $3+2\sqrt[3]{4}+2\sqrt[3]{2}$ (b) $\frac{25-18\sqrt[3]{2}+6\sqrt[3]{4}}{29}$

(c) $\frac{3+\sqrt{15}+2\sqrt{6}}{6}$ (d) $\frac{\sqrt[3]{3}+1}{2}$

5. $\frac{35}{33}$ **7.** 5·99 **8.** 2 **10.** 3·66.

Exercise 12

1. $-(18+11\sqrt{6})$ **2.** $-(142-22\sqrt{35})$

3. $-(24\sqrt{2}+20\sqrt{42}-6\sqrt{6}-15\sqrt{14})$ **4.** $\frac{3-\sqrt{-5}}{14}$

5. $\frac{1}{7}(6+2\sqrt{-5}+3\sqrt{-3}-\sqrt{15})$ **6.** $-\frac{31}{17}i$ **7.** $\frac{4xyi}{x^2+y^2}$

8. $\frac{-6abi}{a^2+4b^2}$ **9.** $0+i$ **10.** $\frac{13}{10}+i\frac{9}{10}$

11. $\frac{18}{13}+i.\frac{1}{13}$ **12.** $\frac{7}{5}+i\frac{\sqrt{6}}{5}$ **13.** $5+i5$ **14.** $65+i.0$

15. $\pm(3+\sqrt{-5})$ **16.** $\pm(5-3\sqrt{-2})$ **17.** $\pm(5-6\sqrt{-1})$

18. $\pm(3i-4)$ **19.** $\pm(3-4i)$ **20.** $\pm\frac{1}{\sqrt{2}}(1-i)$

21. $\pm(3+5i)$ **22.** $\pm\frac{1}{2}(3+\sqrt{-11})$ **23.** $\pm(5+2i)$

24. $\pm(2x+iy)$ **25.** $\pm\frac{1}{\sqrt{2}}(\sqrt{a^2+3a+1}+i\sqrt{a^2-3a+1})$

26. $\pm\left\{\left(a-\frac{1}{a}\right)-2i\right\}$ **27.** $3\sqrt{2}$ **33.** 2 **34.** $\frac{6}{13}$

35. 5 **36.** 13 **37.** 65 **38.** $\frac{25}{13}$ **39.** $\sqrt{\frac{5}{13}}$

40. $(x+y)(x+\omega y)(x+\omega^2 y)$

41. $(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$ **42.** $(x+i)(x-i)$

43. $(x+\omega y)(x+\omega^2 y)$ **45.** ± 1 and $\pm i$ **50.** 3

51. $(a+b+c-abc)^2+(1-ab-ac-bc)^2$ **52.** 10.

Exercise 13

1. $\left.\begin{matrix}x=5\\y=7\end{matrix}\right\}$ or $\left.\begin{matrix}x=7\\y=5\end{matrix}\right\}$ **2.** $\left.\begin{matrix}x=2\\y=5\end{matrix}\right\}$ or $\left.\begin{matrix}x=5\\y=2\end{matrix}\right\}$

3. $\left.\begin{matrix}x=3\\y=2\end{matrix}\right\}$ or $\left.\begin{matrix}x=-1\\y=6\end{matrix}\right\}$ **4.** $\left.\begin{matrix}x=\frac{1}{3}\\y=\frac{1}{2}\end{matrix}\right\}$ or $\left.\begin{matrix}x=\frac{5}{2}\\y=-\frac{5}{3}\end{matrix}\right\}$

5. $x=\frac{3}{5},\ y=\frac{4}{5}$ **6.** $\left.\begin{matrix}x=4\\y=3\end{matrix}\right\}$ or $\left.\begin{matrix}x=-\frac{7}{2}\\y=-\frac{9}{2}\end{matrix}\right\}$

7. $\left.\begin{matrix}x=\frac{3}{2}+\sqrt{2}\\y=\frac{3}{2}-\sqrt{2}\end{matrix}\right\}$ or $\left.\begin{matrix}x=\frac{3}{2}-\sqrt{2}\\y=\frac{3}{2}+\sqrt{2}\end{matrix}\right\}$

8. $\left.\begin{matrix}x=\frac{1}{5}(1+2\sqrt{5a-1})\\y=\frac{1}{5}(2-\sqrt{5a-1})\end{matrix}\right\}$ or $\left.\begin{matrix}x=\frac{1}{5}(1-2\sqrt{5a-1})\\y=\frac{1}{5}(2+\sqrt{5a-1})\end{matrix}\right\}$

9. $\left.\begin{matrix}x=1\\y=9\end{matrix}\right\}$ or $\left.\begin{matrix}x=9\\y=1\end{matrix}\right\}$ **10.** $\left.\begin{matrix}x=\frac{1}{5}\\y=5\end{matrix}\right\}$ or $\left.\begin{matrix}x=\frac{4}{5}\\y=20\end{matrix}\right\}$

11. $\left.\begin{matrix}x=1\\y=3\end{matrix}\right\}$ or $\left.\begin{matrix}x=3\\y=1\end{matrix}\right\}$ **12.** $\left.\begin{matrix}x=5\\y=7\end{matrix}\right\}$ or $\left.\begin{matrix}x=\frac{11}{9}\\y=-\frac{13}{3}\end{matrix}\right\}$

13. $\left.\begin{matrix}x=5\\y=4\end{matrix}\right\}$ or $\left.\begin{matrix}x=4\\y=5\end{matrix}\right\}$ or $\left.\begin{matrix}x=-4\\y=-5\end{matrix}\right\}$ or $\left.\begin{matrix}x=-5\\y=-4\end{matrix}\right\}$

14. $\left.\begin{matrix}x=1\\y=2\end{matrix}\right\}$ or $\left.\begin{matrix}x=-\frac{20}{13}\\y=\frac{37}{13}\end{matrix}\right\}$ **15.** $\left.\begin{matrix}x=2\\y=\pm 2\end{matrix}\right\}$ or $\left.\begin{matrix}x=-2\\y=\pm\frac{1}{2}\end{matrix}\right\}$

16. $\left.\begin{matrix}x=1\\y=2\end{matrix}\right\}$ or $\left.\begin{matrix}x=\frac{1}{2}\\y=1\end{matrix}\right\}$ **17.** $\left.\begin{matrix}x=2\\y=1\end{matrix}\right\}$ or $\left.\begin{matrix}x=-\frac{1}{2}\\y=-\frac{1}{4}\end{matrix}\right\}$

18. $\left.\begin{matrix}x=1\\y=2\end{matrix}\right\}$ or $\left.\begin{matrix}x=-\frac{1}{3}\\y=-\frac{2}{3}\end{matrix}\right\}$

19. $x=\dfrac{1\pm 2\sqrt{5b-1}}{5},\ y=\dfrac{2\pm\sqrt{5b-1}}{5}$ **20.** $\left.\begin{matrix}x=8\\y=4\end{matrix}\right\}$ or $\left.\begin{matrix}x=4\\y=8\end{matrix}\right\}$

21. $x=7,\ y=5$, or, $x=-5,\ y=-7$

22. $\left.\begin{array}{l}x=7\\y=5\end{array}\right\}\ \left.\begin{array}{l}x=5\\y=7\end{array}\right\}\ \left.\begin{array}{l}x=-5\\y=-7\end{array}\right\}\ \left.\begin{array}{l}x=-7\\y=-5\end{array}\right\}$ **23.** $\left.\begin{array}{l}x=9\\y=4\end{array}\right\}$ or $\left.\begin{array}{l}x=4\\y=9\end{array}\right\}$

24. $x=2,\ y=5$ or, $x=-2,\ y=-5$

25. $\left.\begin{array}{l}x=2\\y=1\end{array}\right\}$ or $\left.\begin{array}{l}x=1\\y=2\end{array}\right\}$ **26.** $\left.\begin{array}{l}x=6\\y=10\end{array}\right\}$ or $\left.\begin{array}{l}x=4\\y=15\end{array}\right\}$

27. $\left.\begin{array}{l}x=\frac{1}{2}\\y=1\end{array}\right\}$ or $\left.\begin{array}{l}x=-1\\y=-\frac{1}{2}\end{array}\right\}$ **28.** $\left.\begin{array}{l}x=a\\y=b\end{array}\right\}$ or $\left.\begin{array}{l}x=2a-b\\y=2b-a\end{array}\right\}$

29. $x=\dfrac{b\pm\sqrt{a^2+ab+b^2}}{a+b},\ y=\dfrac{a\mp\sqrt{a^2+ab+b^2}}{a+b}$

30. $\left.\begin{array}{l}x=3\\y=6\end{array}\right\}$ or $\left.\begin{array}{l}x=6\\y=3\end{array}\right\}$ **31.** $\left.\begin{array}{l}x=9\\y=4\end{array}\right\}$ or $\left.\begin{array}{l}x=4\\y=9\end{array}\right\}$

32. $\left.\begin{array}{l}x=0\\y=0\end{array}\right\}$ or $\left.\begin{array}{l}x=2a\\y=2b\end{array}\right\}$ **33.** $\left.\begin{array}{l}x=2\\y=0\end{array}\right\}$ or $\left.\begin{array}{l}x=-12\\y=-\frac{28}{3}\end{array}\right\}$.

TRIGONOMETRY

Exercise 1

1. $60°, \frac{1}{2}$ **2.** $45°, 1$ **3.** $30°, \dfrac{2}{\sqrt{3}}$ **4.** 1 **5.** $-\dfrac{1}{\sqrt{2}}$

6. $\sqrt{3}$ **7.** $-\dfrac{\sqrt{3}}{2}$ **8.** -1 **9.** $-\sqrt{2}$ **10.** -2

11. $-\sqrt{3}$ **12.** $\dfrac{2}{\sqrt{3}}$ **13.** 0 **14.** 1 **19.** $\dfrac{\sqrt{3}}{2}$

20. $\pm\frac{15}{17}$ **21.** 0 **22.** $-\tan 38°$ **23.** $-\cos 30°$

24. $-\sin\frac{\pi}{9}$ **25.** $120°, 300°, -60°, -240°$, **26.** $60°, 120°$.

27. $30°, 150°$ **28.** $30°, 330°$ **29.** $60°, 90°, 270°, 300°$

30. $30°, 150°$ **31.** $45°, 135°, 225°, 315°$ **32.** (i) $\pm\frac{1}{2}$

(ii) $\pm\dfrac{1}{\sqrt{2}}$ (iii) ± 1 **33.** (i) $240°$ (ii) $330°$ (iii) $405°$

34. $\dfrac{2}{\sqrt{3}}$ **35.** -10 **36.** (i) $\sin\theta,\ -\dfrac{\sqrt{3}}{2}$ (ii) 3 **38.** -1.

Exercise 2

1. (i) $\frac{1-\sqrt{3}}{2\sqrt{2}}$ (ii) $2+\sqrt{3}$ (iii) $-(2+\sqrt{3})$

2. $\sin A \cos B \cos C - \cos A \sin B \cos C + \cos A \cos B \sin C$
$+\sin A \sin B \sin C$.

3. $\sqrt{2}$ 4. $\frac{33}{65}$ 5. $-\frac{85}{36}$ 6. $-\frac{278}{29}$ 27. 0 28. 1.

Exercise 3

1. $\sin 7\theta + \sin 3\theta$ 2. $\sin 15\theta + \sin \theta$ 3. $\frac{1}{2}(\cos 6\theta + \cos \theta)$

4. $\cos(2A-B) - \cos(4A+B)$ 5. $\sqrt{3}\cos 20^\circ$

6. $-2\cos 5\theta \sin 2\theta$ 7. $2\cos 5A \cos 2A$

8. $2\sin 8\theta \sin \theta$ 9. 0 10. $\frac{1}{2}$ 11. $\frac{1}{8}$ 12. $\frac{3}{8}$

13. $\frac{1}{\sqrt{3}}$ 29. $4\sin(B+C)\sin(C+A)\sin(A+B)$

30. $\sin(A+B+C) + \sin(A-B-C) + \sin(A+B-C)$
$+\sin(A-B+C)$

32. $4\sin A \sin B \sin C$ 34. $\frac{3}{4}$ 35. $\tan 4A$

36. $\pm\sqrt{\frac{a^2+b^2}{4-a^2-b^2}}$.

Exercise 4

1. $\frac{3}{4}$ 2. $-\frac{23}{27}$ 3. $\frac{1}{7}$

4. (i) 2 : (ii) $\frac{2\tan A}{1+\tan^2 A}$, $\frac{1-\tan^2 A}{1+\tan^2 A}$. 25. $\frac{3}{2}$.

Exercise 5

1. $\frac{1}{4}\{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}\}$, $\frac{1}{4}\{\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}\}$

3. $\frac{\sqrt{5}+1}{8}$ 4. $\frac{3}{4}$ 5. $\frac{1}{2}$ 8. $\frac{16}{305}$, $\frac{49}{305}$ 10. $\frac{1}{16}$

11. $\dfrac{b^2-a^2}{b^2+a^2}$, $\pm\sqrt{\left(\dfrac{4-a^2-b^2}{a^2+b^2}\right)}$

12. $\sin\dfrac{A}{2}=\frac{1}{2}\{-\sqrt{1+\sin A}-\sqrt{1-\sin A}\}$,

$\cos\dfrac{A}{2}=\frac{1}{2}\{-\sqrt{1+\sin A}+\sqrt{1-\sin A}\}$

21. No. $2\sin\dfrac{\theta}{2}=-\sqrt{1+\sin\theta}+\sqrt{1-\sin\theta}$

24. $\dfrac{2}{\sqrt{5}}$, $-\dfrac{1}{\sqrt{5}}$ **25.** $2\sin\frac{1}{2}A=\sqrt{1+\sin A}+\sqrt{1-\sin A}$.

CO-ORDINATE GEOMETRY

Exercise 1

1. (i) 13 (ii) 5 (iii) $4\sqrt{13}$ (iv) 10 (v) $\sqrt{2(a^2+b^2)}$

2. (a) $\sqrt{10}$ (b) 25 (c) 13 (d) 10

(e) $\sqrt{m^2+n^2}$ (f) $2\sqrt{b^2+d^2}$ (g) $(\cos\theta-\sin\theta)\sqrt{2}$

(h) $2\sqrt{a^2+b^2}$ **3.** (i) $(\frac{5}{2}, \frac{7}{2})$ (ii) $(2, -1)$ (iii) $(\frac{7}{2}, -\frac{7}{2})$

4. (a) $(\frac{12}{7}, \frac{2}{7})$ (b) $(0, -\frac{11}{5})$ (c) $(-11, 16)$ (d) $(9, 8)$

5. $(3, 2\frac{2}{3})$ **6** (i) 4 : 5 (ii) 3 : 2 (externally) **7.** 5 or -1

8. 9 or -3 **9.** 7 or 17 **21.** $(\frac{3}{2}, 5)$ **22.** $(3, 4)$ **23.** $3x+y=4$

24. (i) 11, (ii) $8\frac{1}{2}$, (iii) $22\frac{1}{2}$, (iv) -3, (v) $\frac{1}{2}\sin\theta$,

(vi) 0 **25.** $\sqrt{41}$, $(-\frac{1}{3}, \frac{5}{3})$ **26.** $(1, 3)$ **27.** $(3, 4)$

28. $3\sqrt{2}$, 3, 3 **29.** 3 **30.** -6

32. (i) 5·5 sq. units (ii) 9 sq. units (iii) 10 sq. units

(iv) 17 sq. units **33.** $(3, 4)$; 5 **39.** $1\frac{3}{8}$.

Exercise 2

1. $2x-y=4.$ 2. $6x+8y=25$ 3. $2x-3=0.$

4. $2y-3x=4$ 5. $3x^2+4y^2-16y-16x+32=0.$

6. $y^2=2x-1.$ 7. $x^2+y^2-10x-24y=0.$

8. $x^2+y^2=k^2-a^2$ 9. $2x-y+8=0.$

10. $x+y=k.$ 11. $\frac{a}{2x}+\frac{b}{2y}=1.$

Exercise 3

1. (i) 3 (ii) $\frac{1}{7}$ (iii) $\frac{a}{b}$ (iv) -3 (v) 0.

2. (i) $-\frac{3}{2}, (3, 0)$ (ii) $3, (-2, 0)$ (iii) $-1; (0, 0)$
(iv) $2, (-\frac{5}{2}, 0)$ (v) $-\cot\theta, (-r\sin\theta, 0)$.

3. (i) 90° (0, 0) (ii) 45°, (0, 1) (iii) 135°, (0, −5)
(iv) 60°, (0, 2) (v) $\tan^{-1}\frac{3}{4}, (0, \frac{1}{4})$.

4. (i) -3 and $\frac{2}{3}$ (ii) $-\frac{1}{3}$ and $\frac{1}{4}$
(iii) $-r\operatorname{cosec}\theta$ and $r\sec\theta$.

5. (i) $2x+3y=6$ (ii) $4x+5y+20=0$ (iii) $3x-5y=9$
(iv) $5x-9y+13=0$ (v) $bx+ay=a.$
(vi) $b^2x\cos\alpha+a^2y\sec\alpha=ab.$

6. (i) $9x+8y=29$ (ii) $bx-ay+ab=0$

(iii) $\frac{x}{c}\cos\frac{C+D}{2}+\frac{y}{d}\sin\frac{C+D}{2}=\cos\frac{C-D}{2}.$

(iv) $2x-(p_1+p_2)y+2\alpha p_1p_2=0.$

(v) $\frac{x}{h}\cos\frac{\alpha-\beta}{2}-\frac{y}{k}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha+\beta}{2}.$

(vi) $3x+y=5.$ 7. (i) $6x+11y-9=0,$
$11x+2y-71=0$ and $5x-9y+47=0.$

(ii) $x+y=3$, $12x-5y=70$ and $5x-12y+70=0.$
(iii) $2x+5y=0$, $11x-y-57=0$ and $3x-2y=0.$

8. (i) $x\cos 30^\circ + y\sin 30^\circ = 4$

(ii) $x\cos 135^\circ + y\sin 135^\circ = 7$.

(iii) $x\cos 225^\circ + y\sin 225^\circ = 2\sqrt{2}$.

(iv) $-\frac{6}{\sqrt{205}}x + \frac{13}{\sqrt{205}}y = \frac{19}{\sqrt{205}}$.

9. (i) 1 (ii) 2 (iii) $\frac{7}{\sqrt{13}}$ (v) $\frac{ab}{\sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)}}$

10. (a) $y = \sqrt{3}x + 1$ (b) $x\sqrt{3} + y = 10$

(c) $x + y + 2 = 0$. (d) $x - y + 1 = 0$

(e) $x - y + 1 = 0$, (1, 2) (f) $3x - 2y + 30 = 0$

(g) $\frac{x}{a} + \frac{y}{b} = 2$ (h) $x + y = 5$

(i) $x(c-a) + y(d-b) = c^2 + d^2 - a^2 - b^2$.

11. $9x - 2y + 1 = 0$. 12. $2x - 3y = 0$ and $8x - 3y = 0$.

13. $3x + 2y - 12 = 0$ or $x + y = 5$.

14. $x + 10y - 28 = 0$. 16. $x + \sqrt{3}y = \pm 6$.

17. $x - y + 1 = 0$; (1, 2). 18. $\frac{x}{3} - \frac{y}{3} = 1$.

19. (a) Angle $= \tan^{-1}\left(-\frac{4}{5}\right)$, $\frac{3}{5}$ in the negative side of y-axis

(b) $2\sqrt{10}$ (c) 2, $\frac{1}{3}$. 20. $3x + 11y - 15 = 0$.

21. (i) $x'^2 = 4y'$ (ii) $2x'^2 + 3y'^2 - 12x' + 9y' + 14 = 0$.

22. $x'^2 + y'^2 + 4x' + 3y' + 7 = 0$. 23. $x'^2 + 2cx' + y'^2 = 0$.

Exercise 4

1. (a) $y = 7$ (b) $x + 3 = 0$ 2. (i) $\tan^{-1}\frac{1}{7}$

(ii) 45° (iii) 45° (iv) $\tan^{-1}\frac{a^2 - b^2}{2ab}$ (v) 90°.

3. (i) $\left(-\frac{11}{29}, \frac{41}{29}\right)$ (ii) $\left(1\frac{1}{5}, 1\frac{1}{5}\right)$ (iii) $\left(\frac{20}{9}, \frac{20}{9}\right)$; $9x+9y=40$

(iv) $\left(p\cos\frac{\theta+\phi}{2}\sec\frac{\theta-\phi}{2}, p\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}\right)$.

4. (*a*) $4x-3y+3=0$ (*b*) $3x+4y-25=0$

(*c*) $y-3x+7=0$ (*d*) $3x+y-5=0$

(*e*) $3x+7y=0$ (*f*) $12x+18y+11=0$

(*g*) $x+3y-1=0$ (*h*) $x+y+2=0$ (*i*) $x=y$

(*j*) $119x+102y-125=0$ (*k*) $2x+3y+1=0$.

5. (a) $(-2, 1)$ (b) $(13, 7)$ (*c*) $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$

(d) $\left(\frac{d}{a+b+c}, \frac{d}{a+b+c}\right)$ (e) $\left(\frac{15}{17}, \frac{31}{17}\right)$.

6. (a) -1 (b) 0 **10.** $3x-y=7$; $8\frac{1}{6}$ sq. units.

11. $121y-88x=371$ **13.** $\left(\frac{b}{3}, \frac{b}{3}\right)$.

15. (a) $3x-y=7$ and $x+3y=9$

(b) $x=7$ and $x+\sqrt{3}y=7+9\sqrt{3}$

(c) $x=0$ and $y+\sqrt{3}x=0$ (d) $y=x\tan\theta$.

16. $x=3$; $y=4$, $4\frac{1}{2}$ sq. units.

17. $9x-7y-1=0$ and $7x+9y-73=0$.

18. $7x-y+19=0$ and $x-5y+27=0$.

19. (a) $\frac{1}{2}(2+\sqrt{3})$. (b) $\frac{x}{2}-\frac{\sqrt{3}}{2}y=\pm 5$; 10 and $\frac{10\sqrt{3}}{3}$.

20. (a) $\frac{5}{4}$ (b) $\frac{7\sqrt{5}}{3}$ **21.** $5x-12y+16=0$; 13.

22. 24 sq. units **23.** $(0, 0)$, $(10, 0)$, $(7, \sqrt{3})$ and $(3, \sqrt{3})$; $7\sqrt{3}$ sq. units.

24. $x(b-b')-y(a-a')=a'b-ab'$;
$x(b-b')+y(a-a')=ab-a'b'$.

Exercise 5

1. (*a*) on the origin side (*b*) on the non-origin side.

2. (*a*) on opposite sides (*b*) on opposite sides

3. (*a*) $\frac{17\sqrt{13}}{13}$ (*b*) $\frac{3}{13}$ **4.** $\frac{7}{\sqrt{2}}, \frac{21}{\sqrt{29}}, \frac{21}{\sqrt{29}}$.

7. $x-2y+11=0$ **8.** $(3, 0)$ **9.** $(-3, 1)$

10. (*a*) $224x-128y+83=0$ and $16x+28y-203=0$.

(*b*) $y-x+12=0$ and $7y+7x-36=0$.

(*c*) $x(\cos\theta-\cos\phi)+y(\sin\theta-\sin\phi)=p_1-p_2$.

and $x(\cos\theta+\cos\phi)+y(\sin\theta+\sin\phi)=p_1+p_2$.

11. $\frac{11}{5}$ (*b*) $\frac{c_1-c_2}{\sqrt{1+m^2}}$.

12. $\left\{\frac{a}{b}(b\pm\sqrt{a^2+b^2}), 0\right\}$

13. $y=a$ and $4x-3y+3a=0$.

14. $(3l-2m+n)^2=25(l^2+m^2)$. **16.** $6x-4y-3=0$.

APPENDIX

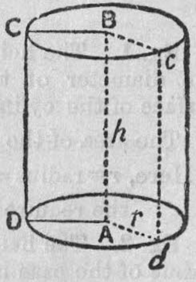
MENSURATION

[**N. B.** Circular cylinder and sphere are included in the syllabus of Core Mathematics, but as questions on these topics may be set in H. S. Elec. Math. papers, they are dealt with here.]

CIRCULAR CYLINDER

A **right circular cylinder** is a solid generated by the revolution of a rectangle round one of its sides as its axis.

A drum, a lead pencil, a heap of pice placed one upon another or a uniformly thick spherical shell, etc. may be taken as circular cylinders.



If the rectangle ABCD is made to turn round the side AB, it generates a right circular cylinder as shown in the figure. AB is said to be the axis of the cylinder. The opposite side CD generates the curved surface of the cylinder and is called the generating line. C and D always remain equidistant from B and A respectively. So, AD and BC describe respectively two plane parallel circles. AB is perpendicular to both the ends. The end on which the cylinder stands is called its base.

Area of the curved surface of a right circular cylinder.

Draw perpendicularly a straight line along the curved surface of a cylinder and cut it along that line. If you now fully stretch the curved surface, it will turn into a plane surface. Evidently this plane surface will be a rectangle whose length and breadth are equal respectively to the circumference and height of the cylinder.

∴ **the area of the curved surface** of the right circular cylinder = circumference of the base × height.

If r be the radius of the base and h the height of a right circular cylinder, then

the **area of the curved surface** of the cylinder

$=$(circumference of the base)$\times$height

$=2\pi rh$ units of area ;

and the **area of the whole surface**

$=$area of the curved surface$+$area of the two ends

$=2\pi rh+2\pi r^2$ ($\because$ area of a circle$=\pi r^2$)

$=2\pi r(h+r)$ units of area.

The **volume** of the cylinder

$=$(area of the base)$\times$height$=\pi r^2 h$ units of volume.

Examples [A]

[Take $\pi=\frac{22}{7}$]

Ex. 1. The height of a right circular cylinder is 1m. 4dm. and the diameter of the base is 5m. Find the area of the curved surface of the cylinder.

The area of the curved surface of the cylinder$=2\pi rh$.

Here, r=radius$=\frac{1}{2}$ diameter $=\frac{5}{2}$ m., h(height)$=$1m. 4dm.$=\frac{7}{5}$m.

$\therefore$ the required area$=2\times\frac{22}{7}\times\frac{5}{2}$ m.$\times\frac{7}{5}$ m.$=22$ sq. metres.

Ex. 2. The height of a cylindrical column is 9 metres and the radius of the base is 1·75 m. Find the area of the whole surface.

Here, radius (r)$=$1·75 m.$=\frac{7}{4}$ m., height (h)$=$9 m.

$\therefore$ the area of the curved surface of the column$=2\pi rh$

$=2\times\frac{22}{7}\times\frac{7}{4}$ m.$\times$9 m.$=$99 sq. m.

Again, the area of the two circular ends$=2\pi r^2$

$=2\times\frac{22}{7}\times(\frac{7}{4})^2$ sq. m.$=19\frac{1}{4}$ sq. m.

$\therefore$ the area of the whole surface$=(99+19\frac{1}{4})$ sq. m.

$=118\frac{1}{4}$ sq. m.$=$118 sq. m. 25 sq. dm.

Ex. 3. Find the radius of the base of a cylindrical column, 8 metres high, whose curved surface is 2464 square metres.

The area of the curved surface of a cylinder$=2\pi rh$.

Here, the given area$=$2464 sq. m. and height (h)$=$8 m.

$\therefore$ $2\pi rh=$2464 sq. m. (r=radius),

or, $2\times\frac{22}{7}\times8$ m.$\times r=$2464 sq. m., $\therefore$ $r=\frac{2464\times7}{2\times22\times8}$ m.$=$49m.

$\therefore$ the required radius$=$49 metres.

Ex. 4. The height of a right circular cylinder is 16 metres and the radius of the base is 3 m. 5 dm. Find its volume.

The volume of the cylinder = area of the base × height = $\pi r^2 h$.

Here, $r = 3\frac{1}{2}$ m. and $h = 16$ m.

$\therefore$ the required volume $= \frac{22}{7} \times (\frac{7}{2})^2 \times 16$ cu. m. = 616 cubic metres.

Ex. 5. The diameter of the base of a cylindrical pillar is 7 metres and its height is 12 metres. Find the cost of constructing it at Rs. $2\frac{1}{3}$ per cubic metre.

Here, $r = \frac{7}{2}$ m., $h = 12$ m., $\therefore$ the volume of the pillar $= \pi r^2 h$

$$= \tfrac{22}{7} \times (\tfrac{7}{2})^2 \times 12 \text{ cu. m.} = 22 \times 7 \times 3 \text{ cu. m.}$$

Cost per cubic metre = Rs. $2\frac{1}{3}$ = Rs. $\frac{7}{3}$.

$\therefore$ the required cost of construction = Rs. $\frac{22 \times 7 \times 3 \times 7}{3}$ = Rs. 1078.

Ex. 6. An iron pipe is 3 inches in bore, $\frac{1}{2}$ inch thick, and 20 feet long. Find its weight supposing that a cubic inch of iron weighs 4·526 ounces. [R. U. S.]

The length of the pipe = 20 feet = 240 inches.

The internal radius of the pipe is $\frac{3}{2}$ inches and the iron plate is $\frac{1}{2}$ inch thick.

$\therefore$ its radius up to the external side $= (\frac{3}{2} + \frac{1}{2})$ or 2 inches.

$\therefore$ if the pipe be solid, the area of its circular end

$$= \tfrac{22}{7} \times 2^2 \text{ sq. inches} = \tfrac{88}{7} \text{ sq. inches.}$$

The area of its hollow circular end $= \frac{22}{7} \times (\frac{3}{2})^2$ sq. inches

$$= \tfrac{99}{14} \text{ sq. inches.}$$

$\therefore$ the volume of the iron plate $= (\frac{88}{7} - \frac{99}{14})$ sq. inches × length

$= \frac{11}{2}$ sq. in. × 240 in. = 1320 cubic inches.

The weight of 1 cubic inch of iron plate = 4·526 ounces.

$\therefore$ the required weight = 4·526 × 1320 ounces

$$= \frac{4{\cdot}526 \times 1320}{16} \text{ pounds} = 373{\cdot}395 \text{ lbs.}$$

Ex. 4 **7.** The curved surface of a right cylinder is 1000 square centimetres and the diameter of the base is 20 cms. ; find the volume of the cylinder. Also find the height to the nearest millimetre. [C. U. 1934]

The circumference of the base $= 2\pi r = 20\pi$ cm.

[$\because$ $2r$ (diameter) = 20 cm.]

$\therefore$ the required height of the cylinder

$$=(1000 \div 20\pi) \text{ cm.} = \frac{50}{\pi} \text{ cm.}$$

$= \dfrac{50}{3{\cdot}1416}$ cm. (taking $\pi = 3{\cdot}1416$) $= 15$ cm. 9 mm. (approximately).

Again, the volume of the cylinder

$$= \pi r^2 h = \pi \times (10)^2 \times \frac{50}{\pi} \text{ c. c.} = 5000 \text{ c. c.}$$

Ex. 8. The curved surface of a cylindrical pillar is 264 square metres and its volume is 924 cubic metres. Find the diameter and the height of the pillar.

Suppose the radius of the pillar $= r$ metres and height $= h$ metres.

$\therefore$ by the condition of the problem,

$2\pi rh = 264 \ldots\ldots (1)$ and $\pi r^2 h = 924 \ldots\ldots (2)$

$\therefore$ from $(2) \div (1)$ we get $\dfrac{\pi r^2 h}{2\pi rh} = \dfrac{924}{264}$, or, $\dfrac{r}{2} = \dfrac{7}{2}$, $\therefore$ $r = 7$.

$\therefore$ the required diameter $= 2r = 14$ metres.

Again, from (i) we get $2\pi \times 7 \times h = 264$, or, $2 \times \frac{22}{7} \times 7 \times h = 264$, or, $44h = 264$, $\therefore$ $h = 6$. $\therefore$ the required height $= 6$ metres.

Ex. 9. How many pieces of coin $\frac{1}{4}$ cm. thick and 2 cm. in diameter can be made by melting a rectangular parallelopiped of metal with dimensions of 22 cm., 6 cm. and 4 cm. ?

The volume of the metallic parallelopiped $= 22 \times 6 \times 4$ cubic cm.

and the volume of each coin $= \pi r^2 h$

$$= \tfrac{22}{7} \times 1^2 \times \tfrac{1}{4} \text{ cu. cm.} = \tfrac{11}{14} \text{ cu. cm.}$$

$\therefore$ the required number of coins $= (22 \times 6 \times 4) \div \frac{11}{14} = 672$.

Ex. 10. A thick hollow cylindrical pipe is 6 inches in length, and its whole surface (outer and inner curved surfaces and the plane edges) is 308 sq. inches. If the external diameter of the pipe is 8 inches, and if its material weighs 4 ozs. per cubic inch, find its weight. [$\pi = \frac{22}{7}$] [H. S. '60]

Let the thickness of the pipe be x inches. As its external radius is 4 inches, its internal radius is $(4 - x)$ inches.

$\because$ by the problem, the area of the outer and inner curved surfaces and the plane edges = 308 sq. in.,

$\therefore\ 2\pi\times 4\times 6+2\pi(4-x)\times 6+2\{\pi.4^2-\pi(4-x)^2\}=308,$

or, $2\pi(24+24-6x)+2\pi(16-16+8x-x^2)=308,$

or, $2\pi(48-6x+8x-x^2)=308,$

or, $\frac{44}{7}(48+2x-x^2)=308$, or, $48+2x-x^2=\frac{308\times 7}{44}=49,$

or, $x^2-2x+1=0$, or, $(x-1)^2=0$, $\therefore\ x=1$ inch.

$\therefore$ The internal radius of the pipe is $(4-1)$ or 3 inches.

$\therefore$ The volume of the material of the pipe

$=(\pi\times 4^2\times 6-\pi\times 3^2\times 6)$ cu. in.

$=\frac{22}{7}\times 6\times(4^2-3^2)$ cu. in. $=\frac{22}{7}\times 6\times 7$ cu. in. $=132$ cu. in.

$\therefore$ The required weight of the pipe = 4 oz. × 132

$=\frac{4\times 132}{16}$ lb. = 33 lbs.

v.9 ✓ **Ex. 11.** A right circular cylinder and a right circular cone have equal bases and equal heights. If their curved surfaces are in the ratio 8 : 5, show that the radius of the base is to its height as 3 : 4. [H. S. '63]

Let the radius of both the cylinder and the cone be r and let h be their height.

The area of the curved surface of the cylinder $=2\pi rh$,

and that of the cone $=\pi r\sqrt{r^2+h^2}$.

$$\therefore \text{ from the given condition } \frac{2\pi rh}{\pi r\sqrt{r^2+h^2}}=\frac{8}{5}, \text{ or, } \frac{h}{\sqrt{r^2+h^2}}=\frac{4}{5},$$

$$\text{or, } \frac{h^2}{r^2+h^2}=\frac{16}{25}, \text{ or, } 9h^2=16r^2, \quad \text{or, } \frac{r^2}{h^2}=\frac{9}{16}, \ \therefore\ \frac{r}{h}=\frac{3}{4}.$$

$\therefore$ The radius of the base : the height = 3 : 4.

v.9 ✓ **Ex. 12.** The volumes of a right circular cylinder and a right circular cone standing on the same base are as 3 : 2. Show that the height of the cone is double the height of the cylinder. [C. Pre-U. '63]

Let h and h' be respectively the heights of the cylinder and the cone and let the radius of their common base be r.

Now, the volume of the cylinder $=\pi r^2 h$

and that of the cone $=\frac{1}{3}\pi r^2 h'$.

$\therefore$ from the given condition we have

$$\frac{\pi r^2 h}{\frac{1}{3}\pi r^2 h'}=\frac{3}{2}, \text{ or, } \frac{h}{\frac{1}{3}h'}=\frac{3}{2}, \text{ or, } \frac{h}{h'}=\frac{3}{2}\times\frac{1}{3}=\frac{1}{2}, \quad \therefore\ h'=2h.$$

$\therefore$ the height of the cone is double the height of the cylinder.

√.?

Ex. 13. A vertical cylindrical vessel of radius 1 foot is partly filled with water, and into it is plunged a solid cone, whose height is equal to the diameter of the base. If, when the cone is completely immersed, the water rises 4 inches, find the dimensions of the cone.

If h be the height of the solid cone, then the radius of its base is $\frac{1}{2}h$.

$\therefore$ The volume of the cone $=\frac{1}{3}\pi(\frac{1}{2}h)^2\times h=\frac{1}{12}\pi h^3$.

Again, when the cone is completely immersed in the water of the cylinder, the water surface rises 4 inches.

Now, the volume of water displaced by the cone $=$ the volume of the cone.

Here, the volume of the water displaced

$=$ the volume of the cylinder whose radius is 1 foot and height is 4 inches $=\pi r^2h=\pi\times(12)^2\times 4$ cu. in.

$\therefore\ \frac{1}{12}\pi h^3=\pi\times 12^2\times 4$, or, $h^3=12^3\times 4$,

$\therefore\ h=12\sqrt[3]{4}$ in. $=12\times 1{\cdot}58$ in. (App.) $=19$ in. (App.)

$\therefore$ The height of the cone $=19$ inches (App.) and the radius of its base $=\frac{1}{2}h=\frac{1}{2}\times 9$ in. $=9{\cdot}5$ inches (App.).

Exercise A

[Take $\pi=\frac{22}{7}$]

1. The length of a hollow right circular cylinder is 10 metres and the diameter of the base is 7 metres. Find the area of its curved surface.

2. The circumference of the base of a cylindrical column is 4 feet 7 inches and its height is 12 yards. Find the area of its curved surface.

3. Find the volume and the area of the curved surface of a right circular cylinder of height 4 feet and radius of whose base is 3 feet. [C. U. '39]

4. The diameter of the ends of a right circular cylinder is 2 m. 8 dm. Find the area of its two ends.

5. The height of a right circular cylinder is 12 cm. and the diameter of the base is 7 cm. Find the area of the whole surface.

6. The height of a cylindrical pillar is 14 m. and its curved surface is 264 square metres. Find the radius of its base.

7. The height of a right circular chimney is 30 ft. and the radius of the base is 1 ft. 2 in. What is the cost of painting its curved surface at 2 annas per square foot ?

8. If it costs Rs. 41. 25 P. to polish the curved surface of a cylindrical pillar 15 metres high at 25 paise per square metre, find the radius of its base.

9. The diameter of a right circular cylinder 14 metres high is 6 metres. Find its volume.

10. The volume of a cylindrical pillar 1 Dm. 4 m. high is 539 cubic metres. Find the diameter of the base.

11. The diameter of the base of a cylindrical pillar is 4 m. and its height is 21 metres. Find the cost of constructing the pillar at $1\dot{6}$ rupees per cubic metre.

✓ 12. A cubic inch of gold is drawn into a wire of 1000 yds. long, find the diameter of the wire to the nearest thousandth of an inch. [C. U. '58]

13. The external and internal radii of the base of a hollow right circular cylinder are 14 cm. and 7 cm. respectively. Find the area of one of its ends.

✓ 14. Find the weight of a cast-iron pipe whose length is 9 feet, the bore 3 inches and thickness of the metal is 1 inch. A cubic inch of cast-iron weighs $\frac{1}{4}$ lb. [R. U. S.]

15. 11 cubic centimetres of iron is drawn into a wire 56 cm. long. Find the radius of the end of the wire.

16. Find the cubic inches of material in a cylindrical tube, the radius of the outer surface being 10 inches, the thickness 2 inches and the height 9 inches. [R. U. S.]

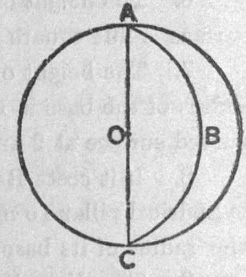
$h = 1000 \times 3 \times 12$ inch

12. $\pi r^2 h = 1$

$r^2 = \frac{7}{22 \times 3 \times 12 \times 1000} = \frac{\sqrt{7}}{(120)\sqrt{55}} = \frac{\sqrt{7} \cdot \sqrt{55}}{6600}$

$19 \cdot 6213$ $\quad \frac{196213}{} = \cdot 002$

A sphere is a solid bounded by one surface and it may be seen to be generated by the revolution of a semi-circle about its diameter as axis. The surface described by the semi-circumference is called the surface of the sphere. The radius of this semi-circle is the radius of the sphere.



A tennis ball, a marble, etc. are familiar examples of a sphere.

If the radius of a sphere be r, then (*a*) the **area of the surface** of the sphere $= \pi \times (\text{ diameter })^2 = 4\pi r^2$ units of area,

or, the area of the surface $=$ the circumference of the generating circle $\times$ diameter $= 2\pi r \times 2r$ units of area.

(*b*) The **volume of the sphere** $= \frac{4}{3}\pi r^3$ units of volume.

Examples [B]

[Take $\pi = \frac{22}{7}$, if not otherwise stated]

Ex. 1. Find the surface and the volume of a sphere whose diameter is 14 metres.

Here, r (radius) $=7$m., $\therefore$ the required area of the surface $= 4\pi r^2 = 4 \times \frac{22}{7} \times (7)^2$ sq. m. $=616$ sq. m.

The required volume $= \frac{4}{3}\pi r^3 = \frac{4}{3} \times \frac{22}{7} \times (7)^3$ cu. m. $= 1437\frac{1}{3}$cu. m.

Ex. 2. The surface of a sphere is 9856 sq. cm. Find its diameter.

$\because$ the area of the surface $= 4\pi r^2$, $\therefore$ here, $4 \times \frac{22}{7} \times r^2 = 9856$ sq.cm.

$\therefore$ $r^2 = \frac{9856 \times 7}{4 \times 22}$ sq. cm. $= 784$ sq. cm., $\therefore$ $r = \sqrt{784}$cm. $= 28$cm.

$\therefore$ the required diameter $= 2r = 56$ cm.

Ex. 3. There are as many cubic inches in the volume of a sphere as there are square inches in the area of its curved surface. Find the radius of the sphere.

Suppose, the radius of the sphere $= r$ inches.

$\therefore$ the area of its surface $= 4\pi r^2$ square inches,

and its volume $= \frac{4}{3}\pi r^3$ cubic inches.

$\therefore$ by the condition of the problem, $\frac{4}{3}\pi r^3 = 4\pi r^2$,

or, $\frac{r^3}{r^2} = \frac{4\pi}{\frac{4}{3}\pi}$, $\therefore$ $r=3$. $\therefore$ the required radius $=3$ inches.

Ex. 4. Find the radius of a sphere whose surface is equal to the curved surface of a right circular cylinder having height and diameter each 10 metres in length.

The radius of the cylinder $= 5$ metres.

The curved surface of the cylinder $= 2\pi rh = 2\pi \times 5 \times 10 = 100\pi$;

Again, the surface of the sphere $= 4\pi r^2$ (taking r for radius of the sphere).

$\therefore\ 4\pi r^2 = 100\pi$, or, $r^2 = 25$, $\therefore\ r = 5$.

$\therefore$ the required radius $= 5$ metres.

Ex. 5. A leaden sphere 1 inch in diameter is beaten inso a circular sheet of uniform thickness equal to $\frac{1}{100}$ inch. Find the radius of the sheet.

The radius of the sphere $= \frac{1}{2}$ inch.

$\therefore$ the volume of the sphere $= \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times (\frac{1}{2})^3 = \frac{1}{6}\pi$ cu. in.

Let the radius of the sheet be r inches.

$\therefore$ the area of the sheet $= \pi r^2$ sq. in. and it is $\frac{1}{100}$ inch thick.

$\therefore$ its volume $= \pi r^2 \times \dfrac{1}{100} = \dfrac{\pi r^2}{100}$ cu. in., $\therefore\ \dfrac{\pi r^2}{100} = \dfrac{\pi}{6}$,

$\therefore\ r^2 = \frac{100}{6} = \frac{50}{3}$ sq. in., $\therefore\ r = \sqrt{\frac{50}{3}}$ in. $= 4{\cdot}0825$ in.

$\therefore$ the required radius of the sheet $= 4{\cdot}0825$ inches (App.).

Ex. 6. How many spherical bullets each 5 dm. in diameter can be cast from a rectangular block of lead 11 m. by 10 m. by 5 m.? $[\pi = \frac{22}{7}]$

The volume of the block of lead $= 11\text{m.} \times 10\text{m.} \times 5\text{m.} = 550$cu. m.

$\therefore$ the total volume of all the bullets $= 560$ cu. m.

The radius of one bullet $= \frac{5}{2}$ dm. $= \frac{1}{4}$ m.

$\therefore$ the volume of 1 bullet $= \frac{4}{3}\pi r^3 = \frac{4}{3} \times \frac{22}{7} \times (\frac{1}{4})^3 = \frac{11}{7 \times 24}$ cu. m.

$\therefore$ the number of bullets $= (550$ cu. m. $\div \frac{11}{7 \times 24}$ cu. m.$) = 8400$.

Ex. 7. A solid sphere of radius 4 cms. is blown into a hollow sphere of uniform thickness, radius of whose external surface is 5 cms. Find the thickness of the hollow sphere. $[\sqrt[3]{61} = 3{\cdot}94]$

The volume of the solid sphere $= \frac{4}{3}\pi r^3 = \frac{4}{3}.\pi.4^3$ cu. cm.

The volume of the hollow sphere $= \frac{4}{3}.\pi.5^3$ cu. cm.

Let the thickness of the hollow sphere be d cm. Then the radius of this sphere is $(5 - d)$ cm.

Now, the volume of this hollow sphere

= the volume of the whole sphere − the volume of the solid portion of the whole sphere.

∴ $\frac{4}{3}\pi\,(5-d)^3=\frac{4}{3}.\pi.5^3-\frac{4}{3}\pi.4^3$,

or, $(5-d)^3=5^3-4^3=125-64=61$, or, $5-d=\sqrt[3]{61}=3{\cdot}94$,

∴ $d=5-3{\cdot}94=1{\cdot}06$. ∴ The reqd. thickness is 1·06 cm.

Ex. 8. The external and internal radii of a sphere are 6 cm. and 3 cm. respectively ; find the volume.

Here, the difference of the volumes of two concentric spheres having 6 cm. and 3 cm. as their respective radius is the volume of the sphere.

∴ the required volume of the sphere $=\frac{4}{3}\pi\times(6)^3-\frac{4}{3}\pi\times(3)^3$
$=\frac{4}{3}\pi(6^3-3^3)=\frac{4}{3}\times\frac{22}{7}\times189$ cu. cm. $=792$ cu. cm.

Ex. 9. The external diameter of a sphere, made of iron sheet 2 inches thick, is one foot. If one cubic foot of iron weighs 450 lbs ; find the weight of the sphere. $[\pi=\frac{22}{7}]$

The external radius of the sphere = 6 inches.

∴ its internal radius = 6 inches − 2 inches = 4 inches.

∴ the volume of the sphere $=\frac{4}{3}\pi(6^3-4^3)=\frac{4}{3}\times\frac{22}{7}\times152$ cu.in.

$=\frac{4\times22\times152}{3\times7\times(12)^3}$ cu. ft. ∵ the weight of 1 cu. ft. of iron = 450lbs.,

∴ the required weight of the sphere

$=\frac{4\times22\times152\times450}{3\times7\times(12)^3}$ lbs. $=\frac{10450}{63}$ lbs. $=165{\cdot}87$ lbs. (approximately).

Ex. 10. Three solid spheres of gold whose radii are 1 cm., 6 cm. and 8 cm. respectively are melted into a single solid sphere. Find the radius of the sphere so formed. [C.Pre-U. '61 ; C.U. '56]

Let R be the radius of the new sphere.

∴ the volume of the new sphere $=\frac{4}{3}\pi R^3$.

Again, the volume of the first sphere $=\frac{4}{3}.\pi.\,1^3$; that of the second sphere $=\frac{4}{3}.\pi.6^3$ and the volume of the third sphere $=\frac{4}{3}.\pi.8^3$.

∴ the total volume of the three spheres $=\frac{4}{3}\pi(1^3+6^3+8^3)$ cu. cm.

$=\frac{4}{3}\pi\times(1+216+512)$ cu. cm. $=\frac{4}{3}.\pi.729$ cu. cm.

∴ $\frac{4}{3}.\pi.R^3=\frac{4}{3}.\pi\times729$, ∴ $R^3=729=9\times9\times9$, ∴ $R=9$.

∴ the required radius of the new sphere = 9 cm.

Ex. 11. How many solid spheres, each of 6 cms. diameter, can be moulded from a solid metal cylinder whose length is 45 cms. and diameter 4 cms.?

If the cylinder of the above dimensions be hollow, how many circular discs of diameter 6 cms. may be made out of it? [C.U. '50]

First Part : The volume of the given cylinder

$=\pi r^2 h=\pi\times(2)^2\times 45$ cu. cm. $=180\pi$ cu. cm.

The volume of each sphere $=\frac{4}{3}\pi r^3=\frac{4}{3}\pi.\,(3)^3$ cu. cm.

$=36\pi$ cu. cm.

$\therefore$ The required number of spheres $=180\pi\div 36\pi=5$.

Second Part : The area of the curved surface of the hollow cylinder $=2\pi rh=4.\pi.45$ sq. cm. $=180\pi$ sq. cm.

The area of each circular disc $=\pi r^2=\pi\times(3)^2$ sq.cm. $=9\pi$ sq.cm.

$\therefore$ The required number of discs $=180\pi\div 9\pi=20$.

Ex. 12. A sphere and a right circular cylinder of the same radius have equal volumes. By what percentage does the diameter of the cylinder exceed its height? [C. U. '51]

Let h be the height of the cylinder and r be the radius of the sphere and the cylinder. The volume of the sphere $=\frac{4}{3}\pi r^3$ and that of the cylinder $=\pi r^2 h$.

$\therefore$ by the problem, we have $\frac{4}{3}\pi r^3=\pi r^2 h$, or, $r=\frac{3}{4}h$.

$\therefore$ the diameter of the cylinder $=2r=\frac{3}{2}h$.

Now, $\frac{3}{2}h$ exceeds h by $\frac{h}{2}$, *i.e.*, by 50%.

$\therefore$ The diameter of the cylinder exceeds its height by 50%.

Ex. 13. A solid sphere 6 inches in diameter is formed into a tube 10 inches in external diameter and 4 inches in length; find the thickness of the tube.

Here the radius of the sphere is 3 inches. Suppose the tube to be x inches thick. Its external radius is 5 inches.

The volume of the sphere $=\frac{4}{3}\pi r^3=\frac{4}{3}\pi\times 3^3$ cu. in. $=4\pi\times 3^2$ cu. in.

The volume of the pipe $=4\pi\{5^2-(5-x)^2\}$ cu. in.

$\therefore\ 4\pi\{5^2-(5-x)^2\}=4\pi\times 3^2$, or, $5^2-(5-x)^2=3^2$,

or, $(5-x)^2=5^2-3^2=4^2$, or, $5-x=4$, $\therefore\ x=1$.

$\therefore$ The required thickness of the tube is 1 inch.

Ex. 14. A solid right circular cylinder, whose height is 9 inches and diameter of the base 4 inches, is deformed into a sphere. Find the surface area of this sphere. [H. S. '61 (Compl.)]

Here the radius of the base of the cylinder is 2 inches.

$\therefore$ Its volume $=\pi r^2 h=\pi.2^2.9$ cu. in. $=36\pi$ cu. in.

Let r inches be the radius of the sphere. Then its volume $=\frac{4}{3}\pi r^3$.

$\therefore$ by the given condition $\frac{4}{3}\pi r^3=36\pi$, or, $r^3=27$, $\therefore r=3$ in.

$\therefore$ The required surface area of the sphere $=4\pi r^2$

$=4\times\frac{22}{7}\times 3^2$ sq. in. $=\frac{36\times 22}{7}$ sq. in. $=113\frac{1}{7}$ sq. inches.

Ex. 15. With the material of a hollow sphere of outer diameter 10 cms. and thickness 2 cms. is made a solid right circular cone of height 8 cms. Find the surface area of its curved surface to the nearest centimetre. $[\pi=\frac{22}{7}]$ [H. S. '61]

Let r cm. be the radius of the base of the cone.

$\therefore$ Its volume $=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi r^2\times 8$ cu. cm.

Again, the external diameter of the sphere is 10 cm. and it is 2 cm. thick.

$\therefore$ its internal diameter $=(10-4)$ cm. $=6$ cm. and so its internal radius is 3 cm. and external radius is $(3+2)$ or 5 cm.

$\therefore$ The volume of the sphere $=\frac{4}{3}\pi\,(5^3-3^3)=\frac{4}{3}\pi\times 98$ cu. cm.

$\therefore$ by the problem $\frac{1}{3}\pi r^2\times 8=\frac{4}{3}\pi\times 98$, or, $r^2=49$, $\therefore r=7$ cm.

Let l cm. be the slant height of the cone.

$\therefore$ $l=\sqrt{r^2+h^2}=\sqrt{8^2+7^2}$ cm. $=\sqrt{113}$ cm.

$\therefore$ The area of its curved surface $=\pi r l$

$=\frac{22}{7}\times 7\times\sqrt{113}$ sq. cm. $=234$ sq. cm. (App.).

Ex. 16. Two solid copper spheres of radii 1 cm. and 3 cms. are melted and a solid right circular cone of height 7 cms. is formed of the material. Find the radius of its base. [H. S. '63 (Compl.)]

The total volume of the two spheres

$=\frac{4}{3}\pi(1^3+3^3)$ cu. cm. $=\frac{4}{3}\pi\times 28$ cu. cm.

Let r cm. be the radius of the base of the cone.

Then its volume $=\frac{1}{3}\pi r^2\times 7$ cu. cm.

$\therefore$ From the given condition we have $\frac{1}{3}\pi r^2\times 7=\frac{4}{3}\pi\times 28$,

or, $\frac{7}{3}r^2=\frac{4}{3}\times 28$, or, $r^2=\frac{4}{3}\times 28\times\frac{3}{7}=16$, $\therefore r=4$.

$\therefore$ The required radius $=4$ cm.

Ex. 17. A sphere of diameter 6 cms. is dropped into a cylindrical vessel partly filled with water. The diameter of the vessel is 12 cms. If the sphere be completely submerged, by how much will the surface of the water be raised? [H. S. '63]

Suppose the water-surface of the vessel rises by h cm. when the sphere is submerged in the vessel.

The volume of the water in the vessel thus raised $= \pi r^2 h$ $= \pi . 6^2 . h$ cu. cm. [$\because r = \frac{1}{2} \times 12$ cm. $= 6$ cm.]

Again, the volume of the sphere $= \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times 3^3$ cu. cm.

$\because$ the water-surface of the vessel has been raised h cm. higher by the displacement of water equal to the volume of the sphere,

$\therefore$ $\pi . 6^2 . h = \frac{4}{3}\pi . 3^3$, or, $36h = 36$, $\therefore$ $h = 1$.

$\therefore$ The surface of the water will be raised by 1 cm.

Exercise B

[Take $\pi = \frac{22}{7}$]

1. Find the surface of a sphere whose diameter is 5 dm. 6 cm.

2. The radius of a sphere is $3\frac{1}{2}$ dm.; find the area of its surface.

3. Find the volume of a sphere having a diameter of 1 dm. 4 cm.

4. The surface of a sphere is 154 sq. cm., find its radius.

5. The surface of a globe is $\frac{11}{28}$ sq. m., find its diameter.

6. The volume of a sphere is $1437\frac{1}{3}$ cu. m., find its radius.

7. A sphere is 36 inches in diameter, find its volume in cubic feet. [P. U.]

8. The volume of a sphere is twice the area of its surface. Find the radius of the sphere. [C. U. '53]

9. The height and diameter of the base of a right circular cylinder are each 6 metres. Find the radius of the sphere whose surface is equal to the curved surface of the cylinder.

10. How many spherical bullets, each 1 dm. in diameter, can be formed from an iron ball whose diameter is 6 decimetres?

11. How may spherical bullets each $\frac{1}{2}$ cm. in radius can be cast from a rectangular block of lead 10 cm. long, 8 cm. broad and $5\frac{1}{2}$ cm. thick ?

12. Three solid golden spherical beads of radii 3, 4 and 5 millimetres are melted into one single solid spherical bead. Find the radius of the single spherical bead. [C.U. '44, C. Pre-U. '63]

13. The external and internal diameters of a shell are respectively $15\frac{1}{2}$ inches and $10\frac{3}{8}$ inches, find the volume. [R. U. S.]

14. An iron sphere, 4 cm. in diameter, is beaten into a circular sheet, $\frac{2}{3}$ cm. thick ; find the radius of the sheet.

15. If r_1 and r_2 be the radii of two solid spheres of gold and if they are melted into one solid sphere, prove that the radius of the new sphere is $\left(r_1{}^3+r_2{}^3\right)^{\frac{1}{3}}$.

16. Find the weight of a hollow iron shell, if the exterior diameter is 13 inches and the thickness of the iron be 2 inches. (Iron weighs 4·2 ozs. per cubic inch.) [R. E.]

17. A lump of clay in the form of a solid sphere is converted into a right circular cylinder of height 16 inches. Find the radius of the base of the cylinder supposing it to be equal to the radius of the sphere. [C. U, '49]

18. How many solid cylinders each of length 8 inches and diameter 6 inches can be made out of the materlal of a solid sphere of radius 6 inches ? [B. U. E. '62, C. U. '52]

Miscellaneous Examples

Ex. 1. The area of the whole surface of a rectangular parallelopiped is 192 sq. cm. and its volume is 144 c. c. If the length of a diagonal be 13 cm., find the dimensions of the solid. [C. U. '57]

Let a, b, c centimetres be respectively the length, breadth and height of the parallelopiped, *i.e.*, $2(ab+bc+ca)=192$ sq. cm.,

$$\therefore \quad ab+bc+ca=96 \ldots(1)$$

Its volume, i.e., $abc=144$ c. c., $\quad \therefore \quad abc=144 \ldots(2)$

The length of the diagonal, i.e., $\sqrt{a^2+b^2+c^2}=13$,

$\therefore \quad a^2+b^2+c^2=169\ldots(3)$

Now, $(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)=169+192=361$

$\therefore \quad a+b+c=\sqrt{361}=19\ldots(4)$

From (1) we have $b(a+c)+ca=96\ldots(5)$

From (2) we have $ca=\dfrac{144}{b}$, and from (4) we have $a+c=19-b$.

$\therefore$ from (5) we have $b(19-b)+\dfrac{144}{b}=96$,

or, $b^2(19-b)+144=96b$, or, $b^3-19b^2+96b-144=0$,

or, $b^3-3b^2-16b^2+48b+48b-144=0$,

or, $b^2(b-3)-16b(b-3)+48(b-3)=0$,

or, $(b-3)(b^2-16b+48)=0$, or, $(b-3)(b-4)(b-12)=0$.

$\therefore$ $b=3, 4, 12$. Now, if $b=3$, then the breadth is 3 cm.

Again, $a+b+c=19$, or, $a+c=19-b=19-3=16$,

and $ac=\dfrac{144}{b}=\dfrac{144}{3}=48$. $\therefore$ $a=12, c=4$.

Hence the reqd. dimensions are 12 cm., 3 cm., 4 cm. (if $b=3$),

or, 12 cm., 4 cm., 3 cm. (if $b=4$).

[If $b=12$, then the length a becomes less than breadth b, so $b=12$ is not considered.]

Ex. 2. The length, breadth and the height of a closed box are 12 in., 10 in. and 8 in. respectively and the total inner surface is 376 sq. inches. If the walls are uniformly thick, find the thickness. [C. U. '58]

Let the thickness be x inches. The inner length, breadth and height of the box are $(12-2x)$, $(10-2x)$ and $(8-2x)$ inches respectively.

$\therefore$ By the problem we have

$2(12-2x)(10-2x)+2(10-2x)(8-2x)+2(8-2x)(12-2x)=376$,

or, $8(6-x)(5-x)+8(5-x)(4-x)+8(4-x)(6-x)=376$,

or, $(6-x)(5-x)+(5-x)(4-x)+(4-x)(6-x)=47$,

or, $30-11x+x^2+20-9x+x^2+24-10x+x^2=47$,

or, $3x^2-30x+27=0$, or, $x^2-10x+9=0$,

or, $(x-1)(x-9)=0$, $\therefore$ $x=1$ or 9.

As the thickness here cannot be 9 inches, the required thickness is 1 inch.

Ex. 3. Show how to draw a plane parallel to the base of a right circular cone so that it divides the cone into

(*i*) two parts of equal surfaces ;

(*ii*) two parts of equal volumes. [C. U. '47]

(*i*) **First Part :** See Example 10 [cone] of this book.

$$\left[\text{ Write up to } \frac{PC^2}{PA^2}=\frac{1}{2}\right] \quad \because \quad \frac{PC^2}{PA^2}=\frac{1}{2}, \quad \therefore \quad \frac{PA^2}{PC^2}=\frac{2}{1},$$

$$\therefore \quad \frac{PA}{PC}=\frac{\sqrt{2}}{1}, \quad \text{or,} \quad \frac{PO}{PO'}=\frac{\sqrt{2}}{1}, \quad \therefore \quad \frac{PO-PO'}{PO'}=\frac{\sqrt{2}-1}{1}.$$

Hence the plane should divide the height of the cone in the ratio $\sqrt{2}-1:1$.

(*ii*) **Second Part :** $\because$ the plane divides the cone into two parts of equal volume,

$$\therefore \quad \tfrac{1}{3}\pi . r^2 . PO = 2 \times \tfrac{1}{3}\pi r'^2 \times PO',$$

$$\therefore \quad \frac{r^2 . PO}{r'^2 . PO'}=2, \quad \text{or} \quad \frac{PO^2}{PO'^2}\cdot\frac{PO}{PO'}=2 \left[\because \quad \frac{r}{r'}=\frac{PO}{PO'}\right]$$

$$\text{or,} \quad \frac{(PO)^3}{(PO')^3}=2, \quad \therefore \quad \frac{PO}{PO'}=\frac{\sqrt[3]{2}}{1}, \quad \therefore \quad \frac{PO-PO'}{PO'}=\frac{\sqrt[3]{2}-1}{1}.$$

Hence the plane will divide the height in the ratio $\sqrt[3]{2}-1:1$.

Ex. 4. The curved surface of a right circular cylinder and a right circular cone standing on the same base and having the same height are in the ratio 8 : 5. Show that the radius of the base is $\frac{3}{4}$ the height. [C. U. '59 (compl.)]

Let r be the radius of the base and h the height of the cylinder.

Then the area of the curved surface of the cylinder $=2\pi rh$, and that of the cone $=\pi rl=\pi r\sqrt{r^2+h^2}$.

$\therefore$ by the problem we have

$$2\pi rh : \pi r\sqrt{r^2+h^2}=8:5, \quad \text{or,} \quad 2h : \sqrt{r^2+h^2}=8:5,$$

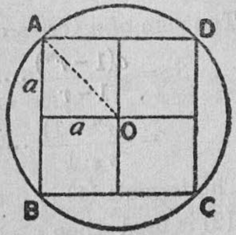
$$\text{or,} \quad 10h=8\sqrt{r^2+h^2}, \quad \text{or,} \quad 100h^2=64r^2+64h^2,$$

or, $64r^2=36h^2$, or, $8r=6h$, $\therefore\ r=\frac{3}{4}h$.

∴ The radius of the base is $\frac{3}{4}$ of the height.

V. 5 **Ex. 5.** The trunk of a tree is a right circular cylinder 5 ft. in radius and 30 ft. high ; find the volume of the timber which remains when the trunk is trimmed just enough to reduce it to a rectangular parallelopiped on a square base. [A. U.]

Here each end of the trunk is a circle whose redius is 5 ft. and its heght is 30 ft.



Let the square *ABCD* be the base of the parallelopiped into which the trunk has been reduced. The square is inscribed in the circle whose centre is *O*.

Here $AO=5$ ft.

$\therefore\ a^2+a^2=AO^2$, or, $2a^2=5^2=25$,

or, $a^2=\frac{25}{2}$, $\therefore\ a=\frac{5}{\sqrt{2}}=\frac{5\sqrt{2}}{2}$ ft.

$\therefore\ AB=2a=5\sqrt{2}$ ft.

∴ The area of the square $=(5\sqrt{2})^2$ sq. ft. $=50$ sq. ft.

∴ the required volume of the timber = area of base × height

$= 50$ sq. ft. × 30 ft. $= 1500$ cu. ft.

volume of the cone $=\frac{1}{3}\pi\cdot(5)^2\cdot 9=75\pi$

lume of
ter in 1 hour

Ex. 6. Water flows at the rate of 20 feet per minute from a cylindrical pipe ·25 inch in diamater. How long would it take to fill a conical vessel, whose diameter at the surface is 10 inches and depth 9 inches ? [S. F. '62 (compl.)] [Ans. 20 mins.]

Area of
cross sec
$\pi r^2=\frac{2}{7\times}$
$r=\frac{25}{100}$
$=\frac{1}{8}$

$\frac{\times 12\times 60\times 22}{7\times 8\times 8}$

e taken
$\frac{7\times 8\times 8\times 75\times 22}{60\times 12\times 22\times 7}$
20 minute.

Ex. 7. A cone, a hemisphere and a cylinder stand on equal bases and have the same height. Show that their volumes are in the ratio 1 : 2 : 3. Compare the whole surfaces. [S. F. '63]

[Ans. Surfaces $\sqrt{2}+1 : 3 : 4$]

Ex. 8. A right circular cone is cut by two planes parllel to the base and trisecting the height. Show that the volumes of the three portions into which the cone is divided are as 1 : 7 : 19. [S. F. '63 (compl.)]

[ALGEBRA]

Summation of an infinite G. P. series

In dealing with Geometrical Progression we have seen that $a+ar+ar^2+ar^3+\ldots\ldots+ar^{n-1}$ is a series in G. P. of n terms. Its first term is a, common ratio is r, number of terms is n and it is a finite series.

The sum of n terms of this series, i.e.,

$$S_n=\frac{a(1-r^n)}{1-r}\ldots\ldots(1)$$

$$=\frac{a(r^n-1)}{r-1}\ldots\ldots(2).$$

The formula-(1) is used when r is numerically less than 1 and (2) is used when $r>1$(numerically).

From formula-(1) we have $S_n=\frac{a}{1-r}-\frac{ar^n}{1-r}$.

Now, if r is numerically less than 1, i.e., if r is a proper fraction, positive or negative, then as the value of n goes on increasing, the absolute value of r^n and therefore of $\frac{r^n}{1-r}$ goes on decreasing. Thus when n is infinitely large, $\frac{r^n}{1-r}$ becomes infinitely small (i.e., almost equal to zero) and then the value of S_n is $\frac{a}{1-r}$.

$\therefore$ When r is numerically less than 1, the sum of the above series to infinity is $\frac{a}{1-r}$,

i.e., $S_\infty=\frac{a}{1-r}$ [S_∞ denotes sum to infinity.]

Example : Sum to infinity the series $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots\ldots$

Here the first term $a=1$, the common ratio $r=\frac{1}{3}\div 1=\frac{1}{3}$;

$$\therefore\quad S_\infty=\frac{a}{1-r}=\frac{1}{1-\frac{1}{3}}=\frac{1}{\frac{2}{3}}=\frac{3}{2}=1\tfrac{1}{2}.$$

Recurring decimal : A recurring decimal can be expressed as an infinite series in G. P. Thus,

(i) $\cdot\dot{7} = \cdot 777\cdots$ to $\infty = \cdot 7 + \cdot 07 + \cdot 007 + \cdots\cdots$ to ∞

$= \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + \cdots\cdots$ to ∞, which is an infinite series in G. P., whose first term is $\frac{7}{10}$ and common ratio is $\frac{1}{10}$.

(ii) $\cdot 1\dot{7}\dot{3} = \cdot 1737373\cdots$ to $\infty = \cdot 1 + \cdot 073 + \cdot 00073 + \cdots$ to ∞

$= \frac{1}{10} + \left(\frac{73}{10^3} + \frac{73}{10^5} + \cdots \text{to } \infty\right)$, the series from the second term is an infinite G. P. series, its first term being $\frac{73}{10^3}$ and the common ratio being $\frac{1}{10^2}$.

Thus a recurring decimal can be expressed as an infinite series in G. P. and its value can be determined by summing the series.

Examples [C]

Ex. 1. Find the sum of $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$ to infinity.

Here the first term $a = 1$, the common ratio $= (-\frac{1}{3}) \div 1 = -\frac{1}{3}$.

$$\therefore \quad S_\infty = \frac{a}{1-r} = \frac{1}{1-(-\frac{1}{3})} = \frac{1}{1+\frac{1}{3}} = \frac{3}{4}.$$

Ex. 2. Sum $\frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \cdots\cdots$ to ∞. [D. U. '22]

Here the given series can be split up into two infinite series in G. P. Thus,

$$S_\infty = \left(\frac{2}{5} + \frac{2}{5^3} + \cdots \text{to } \infty\right) + \left(\frac{3}{5^2} + \frac{3}{5^4} + \cdots \text{to } \infty\right)$$

$$= \frac{\frac{2}{5}}{1 - \frac{1}{5^2}} + \frac{\frac{3}{5^2}}{1 - \frac{1}{5^2}} = \frac{2}{5} \times \frac{25}{24} + \frac{3}{25} \times \frac{25}{24} = \frac{5}{12} + \frac{1}{8} = \frac{13}{24}.$$

Ex. 3. Sum to infinity $(\sqrt{2}+1) + (1) + (\sqrt{2}-1) + \cdots\cdots$

Here $a = \sqrt{2} + 1$ and $r = \frac{1}{\sqrt{2}+1} = \sqrt{2} - 1$.

$$\therefore \quad S_\infty = \frac{a}{1-r} = \frac{\sqrt{2}+1}{1-(\sqrt{2}-1)} = \frac{\sqrt{2}+1}{2-\sqrt{2}} = \frac{(\sqrt{2}+1)(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})}$$

$$= \frac{4+3\sqrt{2}}{2} = 2+\frac{3}{2}\sqrt{2}.$$

Ex. 4. Prove that in a decreasing G.P. continued to infinity, whose common ratio is r, the ratio of any term to the sum of all the succeeding terms is $1-r : r$.

Let a be the first term. Here the common ratio is r.

Let us try with any term, say the pth. term of the series.

Here, the pth. term $= ar^{p-1}$; then the succeeding terms are ar^p, ar^{p+1},......to infinity.

$\therefore$ the sum of all the succeeding terms to infinity $= \dfrac{ar^p}{1-r}$,

$\therefore$ the ratio of the pth. term to the sum of the succeeding terms $= ar^{p-1} : \dfrac{ar^p}{1-r} = ar^{p-1} \times \dfrac{1-r}{ar} = \dfrac{1-r}{r} = 1-r : r$.

Hence the problem is proved.

Ex. 5. Suppose a body moves eternally in this manner, viz., 20 miles in the first minute, 19 miles in the second, $18\frac{1}{20}$ miles in the third and so on in geometrical progression; required the utmost distance it can reach. [C. U. 1864]

Evidently the distance $= (20+19+18\frac{1}{20}+\cdots \text{to } \infty)$ miles.

Here the first term $a=20$, and the common ratio $r=\frac{19}{20}$.

$\therefore$ the required distance $= \dfrac{20}{1-\frac{19}{20}}$ m. $= \dfrac{20}{\frac{1}{20}}$ m. $= 400$ miles.

Ex. 6. Sum $1+3x+5x^2+7x^3+\cdots\cdots$ to infinity $(x<1)$. [C. U. '49]

Let s be the sum. We have

$$\begin{aligned} s &= 1+3x+5x^2+7x^3+\cdots\cdots \text{ to } \infty \\ \therefore\ sx &= \phantom{1+{}} x+3x^2+5x^3+\cdots\cdots \text{to } \infty \quad [\text{multiplying by } x] \\ \hline \therefore\ s(1-x) &= 1+2x+2x^2+2x^3+\cdots\cdots \text{to } \infty \\ &= 1+(2x+2x^2+2x^3+\cdots\cdots \text{to } \infty) \\ &= 1+\frac{2x}{1-x} = \frac{1+x}{1-x}, \quad \therefore\ s = \frac{1+x}{(1-x)^2}. \end{aligned}$$

Ex. 7. If $S_1, S_2, S_3, \cdots\cdots, S_p$ are the sums of infinite geometric series whose first terms are $1, 2, 3, \cdots\cdots, p$ and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\cdots, \frac{1}{p+1}$ respectively, prove that $S_1+S_2+S_3+\cdots\cdots+S_p=\frac{1}{2}p(p+3)$. [B. U. 1888]

From the given conditions we have

$$S_1=1+\frac{1}{2}+\frac{1}{2^2}+\cdots\text{to } \infty=\frac{1}{1-\frac{1}{2}}=2,$$

$$S_2=2+\frac{2}{3}+\frac{2}{3^2}+\ldots\text{to } \infty=\frac{2}{1-\frac{1}{3}}=3,$$

$$S_3=3+\frac{3}{4}+\frac{3}{4^2}+\cdots\text{to } \infty=\frac{3}{1-\frac{1}{4}}=4,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$S_p=p+\frac{p}{p+1}+\frac{p}{(p+1)^2}+\cdots\text{to} \infty=\frac{p}{1-\dfrac{1}{p+1}}=p+1.$$

$\therefore\ S_1+S_2+S_3+\cdots+S_p=2+3+4+\ldots+(p+1)$, where p is the number of terms $=\frac{p}{2}(2+p+1)=\frac{1}{2}p(p+3)$.

Exercise C

Sum the series to infinity :

1. $1-\frac{2}{3}+\frac{4}{9}-\cdots\cdots$

2. $\sqrt{3}+\frac{1}{\sqrt{3}}+\frac{1}{3\sqrt{3}}+\cdots\cdots$

3. $x^2-y^2,\ x+y,\ \frac{x+y}{x-y},\cdots\cdots$

4. $\frac{1}{3}+\frac{1}{7}+\frac{1}{3^2}+\frac{1}{7^2}+\frac{1}{3^3}+\frac{1}{7^3}+\cdots\cdots$

5. Show that $\cdot 1\dot{6}$ is equivalent to an infinite geometric progression. By assuming it find its value. [C. U. '11]

6. Find the sum to infinity of the series $1+\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\cdots$

Show that the sum of the first 10 terms of this series falls short of the sum to infinity by less than a thousand millionth part of 1. [C. U. '35]

7. In a series in G. P. continued to infinity, each term is equal to the sum of all the succeeding terms. Find the series, the first term being unity.

8. The first term of a series in G. P. exceeds the second by 2, and the sum to infinity is 50. Find the series. [C. U.]

9. The sum of an infinite series in G. P. is 3, and the sum of its first two terms is $2\frac{2}{3}$; show that there are two such series. Find them. [C. U. 1934]

10. A body moves in such a manner that it travels a distance of 100 yds. in the first minute, 60 yds. in the second, 36 yds. in the third minute and so on in G. P. Show that the total distance travelled even if the body moves eternally, cannot be greater than 250 yds. [D. B. '31]

[Geometry]

Ex. 1. *Divide a st. line into two parts such that the sum of their squares shall be equal to a given square.*

Let AB be the given st. line and a the side of the given square.

To divide AB into two parts so that the sum of their squares may be equal to a^2.

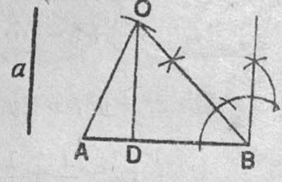


Fig. 53

Construction : At B draw $\angle ABO = 45^\circ$. With centre A and radius a draw an arc of a circle to cut BO at O. Draw $OD \perp AB$. Now AB is divided at D as required.

Proof : $\angle D = 1$ rt. angle, $\angle DBO = 45^\circ$. $\therefore$ $\angle BOD = 45^\circ$. $\therefore$ $DO = DB$. $\because$ $\angle ADO = 1$ rt. angle, $\therefore$ $AO^2 = AD^2 + DO^2 = AD^2 + BD^2$. $\therefore$ $AD^2 + BD^2 = a^2$ [$\because$ $AO = a$]

Ex. 2. *Divide a straight line into two parts so that the square on one part may be twice the square on the other.*

Let AB be the given st. line. To divide AB at P so that $AP^2=2BP^2$. At A draw a right angle and bisect it by AO. Bisect $\angle OAB$ by AC. Now $\angle CAB =22\frac{1}{2}°$. At B draw $BC \perp AB$ cutting AC at C. At C draw $\angle ACP = \angle CAP$ and let CP cut AB at P.

Now, AB is divided as required at P.

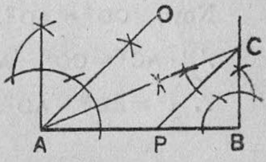


Fig. 54

Proof : In the $\triangle APC$, the ext. $\angle CPB = \angle CAP + \angle ACP =45°$. $\therefore$ $\angle PCB=45°$. $\therefore$ $PB=BC$.

$\because$ $\angle B=1$ rt. angle, $\therefore$ $PC^2=PB^2+BC^2=2PB^2$.

$\therefore$ $AP^2=2PB^2$ ($\because$ $\angle CAP = \angle ACP$, $\therefore$ $AP=CP$).

Ex. 3. *Divide a st. line into two parts so that the square on one part may be three times the square on the other.* [*C.U. 1946*]

To divide the st. line AB at P such that $AP^2=3BP^2$.

At B draw $\angle ABC=45°$ and at A draw $\angle BAC=30°$. Let BC and AC meet each other at C. From C draw $CP \perp AB$. Now, AB is divided as required at P.

Proof : $\angle P=1$ rt. angle, $\angle B=45°$, $\therefore$ $\angle PCB=45°=\angle B$, $\therefore$ $CP=BP$. Again, $\angle A=30°$, $\therefore$ $\angle ACP=60°$, $\therefore$ $AC=2PC$. Now, $AP^2=AC^2-PC^2=(2PC)^2-PC^2=4PC^2-PC^2=3PC^2 =3PB^2$.

Ex. 4. *Divide a given st. line into two parts such that the difference between the squares on the two parts may be equal to the square on a given st. line.*

Let AB be the given st. line and l be the side of the given square.

To divide AB into two parts so that the difference of the squares on the parts is equal to l^2.

Construction : Draw $AC \perp AB$ and make $AC = l$. Join BC. At C draw $\angle BCD = \angle B$ and let CD cut AB at D. Then AB is divided at D as required.

Proof : $\because \angle B = \angle BCD$, $\therefore BD = CD$.

Now, $CD^2 = AD^2 + AC^2$ ($\because \angle A$ is a rt. angle).

$\therefore AC^2 = CD^2 - AD^2 = BD^2 - AD^2$,

$\therefore l^2 = BD^2 - AD^2$ ($\because l = AC$).

ANSWERS

Exercise A

1. 220 sq. m. **2.** 165 sq. ft. **3.** $113\frac{1}{7}$ cu. ft. ; $75\frac{3}{7}$ sq. ft.
4. 1232 sq. dm. **5.** 341 sq. cm. **6.** 3 m. **7.** Rs. 27. 8 as.
8. $1\frac{3}{4}$ m. **9.** 396 cu. m. **10.** 7 m. **11.** Rs. 440
12. ·006 in. **13.** 462 sq. cm. **14.** $339\frac{3}{7}$ lb. **15.** $\frac{1}{4}$ cm.
16. $1018\frac{2}{7}$ cu. in.

Exercise B

1. 98 sq. dm. 56 sq. cm. **2.** 154 sq. dm. **3.** $1437\frac{1}{3}$ cu. cm.
4. $3\frac{1}{2}$ cm. **5.** $\frac{\sqrt{2}}{4}$m. **6.** 7 m. **6.** $14\frac{1}{7}$ cu. ft.
9. 6 units **9.** 3 m. **10.** 216 **11.** 840
12. 6 mm. **13.** 1299·87 cu. in. **14.** 4 cm.
16. 201 lb. $13\frac{3}{5}$ oz. **17.** 12 in. **18.** 4.

Exercise C

1. $\frac{3}{5}$ **2.** $\frac{3\sqrt{3}}{2}$ **3.** $\frac{(x+y)(x-y)^2}{x-y-1}$ **4.** $\frac{2}{3}$
5. $\frac{1}{6}$ **6.** $1\frac{1}{9}$ **7.** $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ to ∞
8. 10, 8, $\frac{32}{5}$,... **9.** One series is 2, $\frac{2}{3}$, $\frac{2}{9}$,$\cdots$, and the other is 4, $-\frac{4}{3}$, $\frac{4}{9}$,......

QUESTIONS

SET IN HIGHER SECONDARY (WEST BENGAL),

C. U. PRE-UNIVERSITY & B. U. ENTRANCE EXAMINATIONS WITH ANSWERS.

[Pertaining to Syllabus for Class X only.]

HIGHER SECONDARY EXAMINATION—1960

Elective Mathematics

First Paper—GROUP A—Algebra

2. Solve the equations :

(*a*) $$\left.\begin{aligned} 2x^2+3xy+y^2&=15 \\ 5x+2y&=12 \end{aligned}\right\} \qquad \left[\text{ Ans. } \begin{aligned} &x=2, \quad y=1 \\ \text{or, } &x=14, \quad y=-29 \end{aligned}\right]$$

(*b*) $$\left.\begin{aligned} 3x+4y&=5xy \\ 2y+3z&=2yz \\ 5z+2x&=6zx \end{aligned}\right\} \qquad \left[\text{ Ans. } \begin{aligned} &x=y=z=0, \\ &x=1,\ y=3,\ z=2 \end{aligned}\right]$$

3. (*a*) A class consists of a number of boys whose ages are in Arithmetical progression, the common difference being 3 months. If the youngest boy is just seven years old and the sum of the ages of the boys is 153 years, find the number of boys in the class. [Ans. 17]

(*b*) If S_1, S_2, S_3 denote respectively the sum of the first n terms, first $2n$ terms and first $3n$ terms of a series in geometrical progression, prove that $S_1(S_3-S_2)=(S_2-S_1)^2$.

4. (*a*) Find the cube roots of unity. If ω be an imaginary cube root of unity, prove that $1+\omega+\omega^2=0$.

$$\left[\text{ Ans. } 1, \ \frac{-1\pm\sqrt{-3}}{2}\right]$$

(*b*) The area of a circle varies as the square of its radius. If the area is $38\frac{1}{2}$ sq. ft. when the radius is 3 ft. 6 in., find the area when the radius is 4 ft. 8 in. [Ans. $68\frac{4}{9}$ sq. ft.]

7. (*a*) Simplify :

$$\log_{10}\frac{384}{5}+\log_{10}\frac{81}{32}+3\log_{10}\frac{5}{3}+\log_{10}\frac{1}{9}. \qquad [\text{ Ans. } 2\ ]$$

(*b*) If x, y, z are in geometrical progression, prove that $\log_{10}x$, $\log_{10}y$ and $\log_{10}z$ are in Arithmetical progression.

Group B—Trigonometry

10. (*a*) If A, B, $A+B$ are all acute angles, prove (geometrically) that $\cos(A+B)=\cos A\cos B-\sin A\sin B$.

(*b*) Find the value of—

$$\sin^2 60^\circ+\cos^2 150^\circ+\tan^2 120^\circ+\cos 180^\circ-\tan 135^\circ.$$

[Ans. $4\frac{1}{2}$]

11. (*a*) Find the values of θ between 0° and 360° which satisfy the equation $2\sin^2\theta+3\cos\theta=0$. [Ans. 120°, 240°]

(*b*) If $A+B=90^\circ$, prove that $\dfrac{\cos 2B-\cos 2A}{\sin 2A}=\tan A-\tan B$.

Second Paper (1960)

2. (*a*) Show that the angle made by a tangent to a circle with a chord drawn from the point of contact is equal to the angle in the alternate segment of the circle.

(*b*) ABC is a triangle inscribed in a circle : AD, AE are lines drawn to the base BC parallel to the tangents at B, C respectively ; prove that $BD : CE = AB^2 : AC^2$.

Or, (*b*) Tangents AB, AC are drawn to a circle ; CE is perpendicular to the diameter BD through B ; prove that AD bisects CE.

3. Draw an equilateral triangle, each side of which is 2 inches. Now proceed to construct a square equal in area to this triangle.

Or, Draw circles of radii 4 cms. and 2·5 cms. respectively, with their centres at a distance 10 cms. apart. Proceed to construct a transverse common tangent to the two circles.

[*Statement of construction, and full, neat and distinct traces are to be given in either case, but no proof.*]

4. Answer either (*a*) and (*b*), or (*c*) and (*d*) :

(*a*) Obtain the co-ordinates of the point which divides the straight line joining the points(x_1, y_1) and (x_2, y_2) internally in the ratio $m_1 : m_2$.

$$\left[\text{Ans. } \frac{m_1x_2+m_2x_1}{m_1+m_2},\ \frac{m_1y_2+m_2y_1}{m_1+m_2}\right]$$

(*b*) If A, B, C, D are points whose co-ordinates are $(-2, 3)$, $(8, 9)$, $(0, 4)$, and $(3, 0)$ respectively, and AB and CD are joined, find the ratio of the segments into which AB is divided by CD.

[Ans. 11 : 47]

(*c*) Obtain the equation of the straight line whose intercepts on the axes OX, OY are a and b respectively.

$$\left[\text{Ans. } \frac{x}{a}+\frac{y}{b}=1\right]$$

(*d*) Determine the equation of the straight line which passes through the intersection of the lines given by $3x-4y+1=0$ and $5x+y=1$, and has equal intercepts of the same sign on the axes.

[Ans. $x+y=\frac{11}{23}$]

7. (*a*) A thick hollow cylindrical pipe is 6 inches in length, and its whole surface (outer and inner curved surfaces and the plane edges) is 308 sq. inches. If the external diameter of the pipe is 8 inches, and if its material weighs 4 ozs. per cubic inch, find its weight. [Take $\pi=\frac{22}{7}$] [Ans. 528 ozs.]

(*b*) When is (*i*) a straight line, (*ii*) a plane said to be perpendicular to a given plane ?

If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, prove that it is perpendicular to the plane containing them.

H. S. Exam. (Compartmental)—1960

First Paper—GROUP A—Algebra

2. (*a*) Solve the equations : $x+2y=4,\ 2xy-y^2=3.$

[Ans. $x=2,\ y=1\ ;\ x=\frac{14}{5},\ y=\frac{3}{5}$]

(*b*) The distance through which a heavy body falls from rest varies as the square of the time of its fall. If a body falls 64 feet in two seconds, how far does it fall in 8 seconds ?

[Ans. 1024 ft.]

3. (*a*) One hundred stones being placed in a straight line on the ground at a distance of one yard from one another, how far will a person travel, who shall bring them, one by one, to a basket, placed in the same straight line at the distance of a yard from the first stone ? [Ans. 5mi. 1300 yds.]

(*b*) If $a,\ b,\ c$ be respectively the p^{th}, q^{th} and r^{th} terms of a geometric series, prove that $a^{q-r}.b^{r-p}.c^{p-q}=1.$

6. (*a*) Find the logarithms of (*i*) 324 to the base $3\sqrt{2}$. (*ii*) $\frac{1}{9}$ to the base $9\sqrt{3}$. [Ans. (*i*) 4 (*ii*) $-\frac{4}{5}$]

(*b*) Prove that $\log\frac{81}{8}-2\log\frac{3}{2}+3\log\frac{2}{3}+\log\frac{3}{4}=0.$

GROUP B—Trigonometry

9. (*a*) If A and B are both acute angles and A is greater than B, prove (geometrically) that

$$\sin(A-B)=\sin A\cos B-\cos A\sin B.$$

(*b*) If $\sin A=\frac{3}{5}$ and $\cos B=\frac{12}{13}$ where A and B are acute angles, find the value of $\dfrac{\tan A-\tan B}{1+\tan A\tan B}$. [Ans. $\frac{16}{63}$]

10. (*a*) Find the values of θ between 0° and 360° which satisfy the equation $\sin^2\theta-2\cos\theta+\frac{1}{4}=0.$ [Ans. 60° and 300°]

(*b*) If $A+B+C=180°$, prove that

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Second Paper (Compl., 1960)

2. (*a*) If two chords of a circle intersect outside the circle, prove that the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other.

(*b*) Prove that if the common chord of two intersecting circles be produced it will bisect their common tangent.

Or, ABC is a triangle right-angled at A ; AD is perpendicular to BC. Show that $AB^2 = BD.BC$.

3. Draw a circle of radius 2 cms. Construct an equilateral triangle circumscribing this circle.

Or, Draw a triangle with sides 3, 4 and 5 cms. Now construct a square equal in area to this triangle.

[*Statement of construction, and full, neat and distinct traces are to be given in either case but no proof*].

4. Answer either (*a*) and (*b*), or (*c*) and (*d*) :

(*a*) Find the distance between the points whose co-ordinates are (x_1, y_1) and (x_2, y_2).

(*b*) Prove that the points whose co-ordinates are $(-2, -2)$, $(2, 2)$ and $(4, -4)$ are the vertices of an isosceles triangle.

(*c*) Find the angle between the straight lines whose equations are $y = m_1x + c_1$ and $y = m_2x + c_2$.

(*d*) Obtain the equation to the straight line passing through the point $(-1, 2)$ and perpendicular to the line $3x + 4y = 5$.

[Ans. $4x - 3y + 10 = 0$]

7. Answer *any two* of the following questions :—

(*a*) Prove that all straight lines drawn perpendicular to a given straight line at a given point of it are coplanar.

(*b*) The volume of a right circular cone whose height is 24 inches is 1232 cu. inches. Find the area of its slant surface.

[Ans. 550 sq. in.]

(*c*) AB is a diameter of a circle ; AC and AD are any two chords cutting the tangent at B in P and Q ; prove that $\angle PCQ = \angle PDQ$.

(*d*) A straight line is drawn through the point $(3, 5)$ such that the point bisects the portion of the line intercepted between the axes. Find the equation of the line, and calculate its perpendicular distance from the origin. [Ans. $5x + 3y = 30$; $\frac{15}{17}\sqrt{34}$]

Higher Secondary Examination, 1961

First Paper

GROUP A—Algebra

2. (*a*) Solve the equations :

$$\left.\begin{array}{r} x+y=3 \\ 2x^2-5xy+2y^2=0 \end{array}\right\} \qquad \left[\begin{array}{ll} \text{Ans.} & x=1,\ y=2\ ; \\ \text{or,} & x=2,\ y=1. \end{array}\right]$$

(*b*) The length of a pendulum varies inversely as the square of the number of beats it makes per minute. If a pendulum 16 ft. long makes 27 beats per minute, find the length of the pendulum that makes 24 beats per minute. [Ans. $20\frac{1}{4}$ ft.]

3. (*a*) A person lends Rs. 1000 to a friend agreeing to charge no interest and also recover the amount by monthly instalments decreasing successively by Rs. 2. In how many months will the loan be paid up, if the first instalment be Rs. 64 and its payment be made one month after the sum is lent ? [Ans. 25]

(*b*) If $1, \omega, \omega^2$ are the three cube roots of unity, prove that

$$(1+\omega-\omega^2)^3=(1-\omega+\omega^2)^3=-8.$$

6. (*a*) Given $\log 2 = \cdot 30103$ and $\log 3 = \cdot 4771213$, find the logarithm of $\cdot 015$. [Ans. $\bar{2}\cdot 1760913$]

(*b*) Prove that $7 \log \frac{10}{9} - 2 \log \frac{25}{24} + 3 \log \frac{81}{80} - \log 2 = 0$.

GROUP B—Trigonometry

9. (*a*) If A, B and $A-B$ are positive acute angles, prove geometrically that $\sin (A-B) = \sin A \cos B - \cos A \sin B$.

(*b*) Find the value of $\sin 330^\circ + \tan 45^\circ - 4 \sin^2 120^\circ + 2 \cos^2 135^\circ + \sec^2 180^\circ$. [Ans. $-\frac{1}{2}$]

10. (*a*) Find the values of θ between 0° and 360° which satisfy the equation $\sqrt{3} \sin \theta + \cos \theta = 1$. [Ans. $0^\circ, 120^\circ, 360^\circ$]

(*b*) $A+B+C=180^\circ$, prove that

$$\tan A + \tan B + \tan C = \tan A . \tan B . \tan C.$$

Second Paper (1961)

2. (*a*) If from a point outside a circle, a secant and a tangent be drawn to the circle, prove that the rectangle contained by the segments of the secant is equal to the square on the tangent.

(*b*) If the diagonals of a cyclic quadrilateral are at right angles, show that the perpendicular from the point of intersection to any side when produced backwards bisects the opposite side.

Or, (*b*) From the extremities of any chord AB of a circle, perpendiculars AQ, BR are drawn to the tangent at any point P. If PM is perpendicular to AB, prove that $PM^2 = AQ.BR$.

3. Draw a circle of radius 1 inch, and then construct a regular hexagon circumscribing the circle.

Or, Take a straight line of length 2 inches and divide it into two parts such that the square on one part may be double the square on the other part. [*Statement of construction and distinct traces are to be given in either case, but no proof.*]

4. Answer either (*a*) and (*b*), or (*c*) and (*d*) :

(*a*) Obtain the area of the triangle whose vertices are points (x_1, y_1), (x_2, y_2) and (x_3, y_3).

[Ans. $\frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$]

(*b*) Find the area of the triangle whose vertices A, B, C are respectively $(3, 4)$, $(-4, 3)$ and $(8, -6)$; hence or otherwise find the length of the perpendicular from A on BC.

[Ans. 37·5 units of area ; 5 units of length]

(*c*) Obtain the equation of the straight line passing through the points (x_1, y_1) and (x_2, y_2). $\left[\text{Ans. } \dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}.\right]$

(*d*) Obtain the equation to the perpendicular bisector of the line joining the points $(-2, 7)$ and $(8, -1)$. At what distance is the perpendicular bisector from the origin ?

[Ans. $5x - 4y - 3 = 0$, distance$= \dfrac{3}{\sqrt{41}}$ units of length]

7. (*a*) A and B are two fixed points whose co-ordinates are (2, 4) and (2, 6) respectively ; ABP is an equilateral triangle on the side of AB opposite to the origin. Find the co-ordinates of P. [Ans. $(2+\sqrt{3}, 5)$]

(*c*) With the material of a hollow sphere of outer diameter 10 cms. and thickness 2 cms. is made a solid right circular cone of height 8 cms. Find the surface area of its curved surface to the nearest square centimetre $[\pi=\frac{22}{7}]$. [Ans. 234 sq. cm.]

(*d*) How is the angle between two intersecting planes defined ? When is a plane perpendicular to another plane ?

If two straight lines are parallel, and if one of them is perpendicular to a plane, prove that the other is also perpendicular to the same plane.

H. S. Exam. (Compartmental)—1961

First Paper

GROUP A—Algebra

2. (*a*) Solve the equations : $\frac{x}{y}+\frac{y}{x}=2\frac{1}{2}$; $x+y=6$.

[Ans. $x=4, y=2$, or, $x=2, y=4$]

(*b*) If x varies as y^2 and $y=4$ when $x=8$, find y when $x=32$.

[Ans. $y=\pm 8$]

3. (*a*) If a, b, c be in Arithmetical Progression and x, y, z in Geometrical Progression, prove that $x^{b-c}y^{c-a}z^{a-b}=1$.

(*b*) If $x=3+4i$ and $y=3-4i$, where $i=\sqrt{-1}$, find the value of x^3+y^3. [Ans. -234]

6. (*a*) If $\frac{\log x}{y-z}=\frac{\log y}{z-x}=\frac{\log z}{x-y}$, prove that $xyz=1$.

(*b*) Show that $\log_{10} 2+16 \log_{10} \frac{16}{15}+12 \log_{10} \frac{25}{24}+7 \log_{10} \frac{81}{80}=1$.

GROUP B—Trigonometry

9. (*a*) If A, B and $A-B$ are all positive acute angles, prove geometrically that $\cos (A-B)=\cos A \cos B+\sin A \sin B$.

10. (*a*) Prove that $\cos 3A=4 \cos^3 A-3 \cos A$.

(*b*) If $A+B+C=180°$, prove that

$$\sin 2A+\sin 2B+\sin 2C=4 \sin A \sin B \sin C.$$

Second Paper (Compl. 1961)

2. (*a*) Prove that the obtuse angle between the tangent at a point of a circle and a chord through the point of contact is equal to the angle in the alternate segment.

Or, If from any point on the circumcircle of a triangle, perpendiculars are drawn to the sides of the triangle, prove that the feet of the perpendiculars are collinear.

(*b*) If two circles intersect, show that their common tangent subtends supplementary angles at the points of intersection.

Or, Two radii of a circle are perpendicular to each other, and a tangent cuts them when produced ; prove that the other tangents drawn to the circle from these points of intersection are parallel.

3. Take a straight line of length 6 cms. ; divide it into two segments such that the rectangle contained by the segments may be equal to a square on a side of length 2 cms.

Or, Draw a circle of radius 1 inch. Find out a point outside this circle such that the two tangents from it to the circle, and the line joining the points of contact may form an equilateral triangle. [*Statement of construction, and full, neat and distinct traces are to be given in either case, but no proof.*]

4. Answer either (*a*) and (*b*) or (*c*) and (*d*) :—

(*a*) Obtain the distance between the points whose rectangular cartesian co-ordinates are (x_1, y_1) and (x_2, y_2).

(*b*) Show that the triangle whose vertices are the points $(-2, -5)$, $(4, -1)$ and $(-1, 0)$ is isosceles.

(*c*) Obtain the equation to a straight line which is inclined to the x-axis at an angle θ, and whose intercept on the y-axis is c.

(*d*) Show that the points $(1, 4)$, $(3, -2)$, and $(-3, 16)$ are collinear.

7. Answer *any two* of the following questions :—

(*a*) A and B are two fixed points on a plane, and a point P moves on the plane in such a way that $PA=2PB$ always. Prove either geometrically or analytically that the locus of P is a circle.

Group C—Solid Geometry and Mensuration

11. Same as Q. 7. (*a*) of H. S. 1960 (Compl.)

12. If a right angle rotates about one of its arms, then the other arm describes a plane.

13. Find the volume and the lateral surface of a right prism 8 inches long, standing on an isosceles triangle, each of whose equal sides is 5 inches and the other side 6 inches.

[Ans. vol.$=96$ cu. in., surface$=128$ sq. inch.]

14. A right pyramid stands on a rectangular base whose sides are 12 inches and 9 inches ; and the length of each of the slant edges is 8·5 inches. Find the height and the volume of the pyramid. [Ans. $h=4''$, $v=144$ cubic inches.]

H. S. Exam. (Compartmental)—1962

First Paper—Group A—Algebra

2. (*a*) Solve the equations :

$$\left.\begin{aligned} x-y&=2\\ x^2+y^2&=34 \end{aligned}\right\} \qquad \left[\text{Ans. } \left.\begin{aligned} x&=5\\ y&=3 \end{aligned}\right\} \text{ or, } \begin{aligned} x&=-3\\ y&=-5 \end{aligned}\right]$$

(*b*) When a body falls from rest, its distance from the starting point varies as the square of the time elapsed. If a body falls from rest through 402½ ft. in 5 seconds, how far does it fall in 10 seconds ?

[Ans. 1610 ft.]

3. (*a*) The fifth term of a G. P is 81 and the second term is 24 ; find the series. [Ans. 16, 24, 36, 54, 81,......]

(*b*) If $i=\sqrt{-1}$, show that $\dfrac{3+2i}{2-5i}+\dfrac{3-2i}{2+5i}=-\dfrac{8}{29}$.

6. (*a*) Find the logarithms of (i) 5832 to the base $3\sqrt{2}$ (ii) 81 to the base $\sqrt[3]{9}$. [Ans. (i) 6, (ii) 6.]

(*b*) Show that $7\log\frac{16}{15}+5\log\frac{25}{24}+3\log\frac{81}{80}=\log 2$.

Group B—Trigonometry

9. (*a*) Same as Q. 9. (*a*) of H. S. 1961.

(*b*) Show that $\cos A+\sin(270^\circ+A)-\sin(270^\circ-A)+\cos(180^\circ+A)=0$.

10. (*a*) Find the values of θ between 0° and 360°, which satisfy the equation $\cot\theta+\tan\theta=2\operatorname{cosec}\theta$. [Ans. 60° or 300°]

(*b*) Prove that $\dfrac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta}=4\cos 2\theta\cos 4\theta$.

Second Paper (Compl. 1962)

Group A—**Plane Geometry**

2. (*a*) Prove that the angles made by a tangent to a circle with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle.

(*b*) *AB* is the common chord of two circles, one of which passes through *O*, the centre of the other : prove that *OA* bisects the angle between the common chord and the tangent to the first circle at *A*.

5. Construct a square equal in area to a given rectangle whose adjacent sides are 1·5 in. and 2·5 in. Measure the side of the square. [*Statement of the construction and the traces are to be given.*]

GROUP B—**Co-ordinate Geometry**

6. (*a*) Same as Q. 4. (*a*) of H.S. 1960 (Compl.)

(*b*) Show that the straight line joining the two points $(-7, 3)$ and $(14, -6)$ passes through the origin.

7. (*a*) Same as Q. 4. (*c*) of H.S. 1961.

(*b*) Show that the three lines $3x+y=5$, $x+5y+3=0$ and $5x-2y=12$ meet in a point.

GROUP C—**Solid Geometry and Mensuration**

11. If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, prove that it is perpendicular to the plane in which they lie.

12. From *O*, the centre of a circle, a perpendicular *OA* is erected to the plane of a circle. Prove that all points on the circumference are equidistant from any point on the perpendicular *OA*.

13. The length, breadth and height of a rectangular block are in the ratio $4:3:2$, and the whole surface of the block is 1872 sq. in. Find the dimensions of the block and its volume.

[Ans. 24″, 18″, 12″ ; 5184 cubic inches]

14. Find the curved surface and the volume of a right circular cylinder whose height is 8 in. and the radius of whose base is 5 in. [$\pi=\frac{22}{7}$]. [Ans. $251\frac{3}{7}$ sq. in. ; $628\frac{4}{7}$ cubic inches.]

H. S. Examination—1963

First Paper—GROUP A—Algebra

2. (*a*) Solve the equations : $x+\dfrac{4}{y}=1$, $y+\dfrac{4}{x}=25$.

[Ans. $x=\frac{1}{5}$, $y=5$; or, $x=\frac{4}{5}$, $y=20$]

(*b*) The volume of a pyramid varies jointly as its height and the area of its base ; and when the area of the base is 60 square feet and the height 14 feet, the volume is 280 cubic feet. What is the area of the base of a pyramid whose volume is 390 cubic feet and whose height is 26 feet ? [Ans. 45 sq. ft.]

3. (*a*) If a, b, c, d be in *G. P.*, show that

$$(b-c)^2+(c-a)^2+(d-b)^2=(a-d)^2.$$

(*b*) If $1, \omega, \omega^2$ be the three cube roots of unity, prove that

(i) $1+\omega+\omega^2=0$, (ii) $(3+3\omega+5\omega^2)^6=(3+5\omega+3\omega^2)^6=64$.

6. (*b*) Given $\log 2=\cdot30103$ and $\log 3=\cdot4771213$, find (i) $\log 75$ and (ii) $\log 4500$. [Ans. (i) 1·8750613 (ii) 3·6532126]

GROUP B—Trigonometry

8. Same as Q. 9(*a*) of H.S. 1961 (Compl.)

9. (*a*) Find the values of θ between 0° and 360° which satisfy the equation $\cos\theta+\sqrt{3}\sin\theta=\sqrt{2}$. [Ans. 15° and 105°]

(*b*) If $A+B+C=180°$, prove that

$$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Second Paper—1963

GROUP A—Plane Geometry

3. (*a*) Show that the acute angle made by a tangent to a circle with a chord drawn from the point of contact is equal to the angle in the alternate segment of the circle.

(*b*) Two circles intersect at A and B, and through P, any point on one of them, straight lines PAC and PBD are drawn to cut the other at C and D. Show that CD is parallel to the tangent at P.

8. Construct, to the scale, an isosceles triangle with each of the equal sides equal to 2 inches, and each base angle double the vertical angle.

Or, Divide a straight line of length 2 inches into two parts such that the square on one part may be three times the square on the other.

[*Statement of construction and full, neat traces are to be given in any one of the above cases, but no proof.*]

GROUP B—Co-ordinate Geometry

5. (*a*) Obtain the distance between two points whose rectangular cartesian co-ordinates are (x_1, y_1) and (x_2, y_2).

(*b*) Prove that the points $(2, -2)$, $(8, 4)$, $(5, 7)$ and $(-1, 1)$ are the successive angular points of a rectangle.

6. (*a*) Obtain the perpendicular distance from the point (x_1, y_1) to the straight line $ax+by+c=0$.

(*b*) Find the ortho-centre of the triangle whose angular points are $(2, 7)$, $(-6, 1)$ and $(4, -5)$. [Ans. $(-\frac{10}{9}, \frac{49}{27})$]

GROUP C—Solid Geometry and Mensuration

10. (*a*) Same as Q. 7(*b*) Second part of H. S. 1960.

(*b*) If $PA=PB=PC$, where P is a point outside the plane of the triangle ABC, and if PO be drawn perpendicular to the plane, prove that O is the circum-centre of the triangle ABC.

(*c*) If two straight lines are both perpendicular to a plane, show that they are parallel.

(*d*) If the middle points of the adjacent sides of a skew quadrilateral are joined, prove that the figure so formed is a parallelogram.

11. A right circular cylinder and a right circular cone have equal bases and equal heights. If their curved surfaces are in the ratio 8 : 5, show that the radius of the base is to the height as 3 : 4.

Or, A sphere of diameter 6 cms. is dropped into a cylindrical vessel partly filled with water. The diameter of the vessel is 12 cms. If the sphere be completely submerged, by how much will the surface of the water be raised? [Ans. 1 cm.]

H. S. EXAMINATION (Compartmental)—1963

First Paper

GROUP A—Algebra

2. (*a*) Solve the equations :

$$\left.\begin{array}{l} x+3y=2 \\ x^2+2y^2+3xy=0 \end{array}\right\}$$

[Ans. $x=-1, y=1$, or $x=-4, y=2$]

(*b*) If the volume of a cone whose height is 12 inches and base 30 sq. inches be 120 cubic inches, find the volume of another cone whose height is 20 inches and base one square foot, the volume of a cone varying as the height and the base jointly.

[Ans. 960 cu. in.]

3. (*a*) If S be the sum, P the product and R the sum of the reciprocals of n terms in $G. P.$,

prove that $P^2=\left(\frac{S}{R}\right)^n$.

(*b*) Find the three cube roots of unity. If $1, \omega, \omega^2$ are the cube roots, prove that

$$(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)=a^3+b^3+c^3-3abc.$$

GROUP B—Trigonometry

8. (*a*) Same as Q. 10. (*a*) of H. S., 1960

(*b*) Simplify :

$$\frac{\sin(B-C)}{\cos B\cos C}+\frac{\sin(C-A)}{\cos C\cos A}+\frac{\sin(A-B)}{\cos A\cos B}$$ [Ans. 0]

9. (*a*) Find the value of θ between 0° and 360° which satisfy the equation $2\sin^2\theta+\sqrt{3}\cos\theta+1=0$. [Ans. 150°, 210°]

(*b*) If $A+B+C=180°$, prove that

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1.$$

Second Paper—1963 (Compl.)

GROUP A—Plane Geometry

3. (*a*) If from any point outside a circle, two secants are drawn to the circle, prove that the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other.

(*b*) *ABCD* is a quadrilateral inscribed in a circle, and the diagonal *BD* bisects *AC* : shew that *AB.AD*=*BC.CD*.

4. (*a*) Draw two circles of radii 1 cm. and 2 cms. with their centres 5 cms. apart, and construct a direct common tangent to these circles.

Or, Draw an equilateral triangle each of whose sides is 4 cms. in length, and then construct a square equal in area to this triangle.

[*Statement of construction and full, neat traces are to be given in any one of the above cases. No proof is necessary.*]

GROUP B—Co-ordinate Geometry

5. (*a*) Same as Q. 4(*a*) of H. S., 1961.

(*b*) Show that the line joining (−4, −5) and (9, 8) bisects the line joining (2, 1) and (6, 5).

6. (*a*) Same as Q. 4(*c*) of H. S., 1961.

(*b*) Find the equation to the st. line which passes through the point $(-5, -8)$ and has equal intercepts of opposite signs on the axes. [Ans. $x-y=3$]

GROUP C—Solid Geometry & Mensuration

10. How is the angle between two intersecting planes defined? When is a plane said to be perpendicular to another plane?

Show that if a straight line is perpendicular to a plane, then any plane passing through the st. line is perpendicular to that plane.

Or, If PN be drawn perpendicular to a plane XY from an outside point P, and from the foot N of the perpendicular, a line NM is drawn perpendicular to the st. line AB in the plane XY, prove that PM is perpendicular to AB.

11. Two solid copper spheres of radii 1 cm. and 3 cms. are melted, and a solid right circular cone of height 7 cms. is formed of the material. Find the radius of its base. [Ans. 4 cm.]

Or, The external length, breadth and height of a closed box are 10 cms., 9 cms., 7 cms. respectively, and the total inner surface is 262 sq. cms. If the walls of the box be uniformly thick, find the thickness. [Ans. 1 cm.]

HIGHER SECONDARY EXAMINATION—1964

FIRST PAPER

GROUP A—Algebra

2. (*a*) Solve the equations:

$$2x-3y=4$$
$$\frac{1}{x}+\frac{1}{y}=\frac{7}{10}$$

[Ans. $x=5,\ y=2$; or, $x=\frac{4}{7}$, $y=-\frac{20}{21}$]

(*b*) Given that the illumination from a source of light varies as the square of the distance, how much farther from a candle must a book, which is now 8 inches off, be removed so as to receive just half as much light? [Ans. $8(\sqrt{2}-1)$ in.]

3. (*a*) A man arranges to pay off a debt of £3600 by 40 annual instalments which form an arithmetical series. When 30 of these instalments have been paid, he dies leaving a third of his debt unpaid ; find the value of the first instalment. [Ans. £51]

(*b*) If ω is an imaginary cube root of unity, prove that
$$(1-\omega^2)(1-\omega^4)(1-\omega^8)(1-\omega^{10})=9.$$

7. (*a*) Given $\log_{10} 165 = 2{\cdot}2175$ and $\log_{10} 6974 = 3{\cdot}8435$, find the value of $\sqrt[5]{{\cdot}00000165}$. [Ans. ·06974]

GROUP B—**Trigonometry**

8. (*a*) Same as Q. 10(*a*) of H. S., 1960.

(*b*) If $\tan\theta+\sec\theta=x$, prove that $\sin\theta=\dfrac{x^2-1}{x^2+1}$.

9. (*a*) Find the values of θ between 0° and 360° which satisfy the equation $3(\sec^2\theta+\tan^2\theta)=5$. [Ans. 30°, 150°, 210°, 330°]

(*b*) If $A+B+C==180^\circ$, prove that
$$\sin(B+2C)+\sin(C+2A)+\sin(A+2B)$$
$$=4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}.$$

Second Paper—1964

GROUP A—**Plane Geometry**

1. Construct a square equal in area to a given rectangle.

Or, Construct a regular hexagon about a given circle.

[*Traces of construction only are required in either of the two constructions.*]

4. Same as Q. 2(*a*) of H. S., 1962.

In a $\triangle ABC$, perpendiculars AP and BQ are drawn from A and B to opposite sides and intersect at O.

Prove that $AO.OP=BO.OQ$.

GROUP B—**Co-ordinate Geometry**

5. Find the co-ordinates of the point which divides the st. line joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio $m : n$.

Write down the co-ordinates of the middle point of the st. line joining the points $(7, -4)$ and $(-5, 6)$. [Ans : $(1, 1)$]

6. Find the equation of the straight line passing through the intersection of the st. lines $2x-7y+11=0$ and $x+3y-8=0$, if it

(*a*) passes through the origin. [Ans : $27x-23y=0$]

(*b*) is perpendicular to the st. line $2x-5y+6=0$.
[Ans : $5x+2y-13=0$]

(*c*) makes equal intercepts on the two axes.
[Ans : $13x+13y-50=0$]

GROUP C—Solid Geom. and Mensuration

10. Give instances from the sides and edges of a cube of :

(*a*) parallel planes, (*b*) planes perpendicular to one another, (*c*) lines parallel to a plane, (*d*) lines perpendicular to a plane, (*e*) pairs of skew lines.

Or, Same as Q. 11 of H. S., 1962 (Compl.).

11. The volume of a right prism is 80 cu. ft. and its base is a triangle whose sides are 3 ft., 4 ft. and 5 ft. respectively. Find the height and the area of the total surface of the prism.

[Ans : height$=13\frac{1}{3}$ ft., area$=172$ sq. ft.]

Or, A conical tent is required to accommodate 4 people ; each person must have 20 sq. ft. of space on the ground and 100 cu. ft. of air to breathe. Find the height and radius of the tent. [$\pi=\frac{22}{7}$]. [Ans : height$=15$ ft., radius$=5{\cdot}05$ ft.]

H. S. Exam. (Compartmental), 1964

First Paper—GROUP A—Algebra

2. (*a*) Solve the equations :

$$\left.\begin{array}{r}\sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}=\dfrac{5}{2}\\ x+y=10\end{array}\right\}$$

[Ans : $x=8,\ y=2$; or, $x=2,\ y=8$]

(*b*) Assuming that the area of a triangle varies as the altitude and base jointly, and that when the altitude is 18 ft. and base

33 ft. 2 in., the area is $33\frac{1}{6}$ sq. yds. ; find the area of a triangle whose base is 10 ft. 4 in. and altitude 2 ft. 9 in.

[Ans : 1 sq. yd. 5 sq. ft. 30 sq. in.]

3. (*b*) If s_1, s_2, s_3 are the sums of n terms, $2n$ terms and to infinity of a series in G. P., prove that $s_1(s_1 - s_3) = s_3(s_1 - s_2)$.

(*b*) Prove that $\left\{\frac{1+\sqrt{-3}}{2}\right\}^6 = 1$.

7. (*a*) Given $\log 2 = \cdot 3010300$ and $\log 3 = \cdot 4771213$, obtain the logarithms of (i) $\cdot 0054$ and (ii) $(\cdot 405)^{\frac{1}{6}}$.

[Ans : (*i*) $\bar{3}\cdot 7323939$, (*ii*) $\bar{1}\cdot 9345759$]

GROUP B—**Trigonometry**

8. (*a*) Same as Q. 9(*a*) of H. S., 1960 (Compl.).

(*b*) If $\tan\theta + \sin\theta = m$, $\tan\theta - \sin\theta = n$, prove that

$$m^2 - n^2 = 4\sqrt{mn}.$$

9. (*a*) Find the values of θ between 0° and 360° which satisfy the equation $\cot\theta + \tan\theta = 2\sec\theta$.

[Ans. : $\theta = 30^\circ, 90^\circ, 150^\circ, 270^\circ$]

(*b*) If $A+B+C = 180^\circ$, prove that

$$\sin(B+C-A) + \sin(C+A-B) + \sin(A+B-C) = 4\sin A\sin B\sin C.$$

Second Paper, 1964 (Compl.)

GROUP A—**Plane Geometry**

1. Draw a mean proportional between two given st. lines. (*Traces and statement only are required but no proof.*)

Or, Draw a direct common tangent to two given circles. (*Traces and statement only are required but no proof*).

2. Same as Q. 3 (*a*) of H. S., 1963.

A triangle ABC is inscribed in a circle and the tangents to the circle at A, B, C form a triangle PQR, QR being the tangent at A and RP the tangent at B. If the angle $P = 54^\circ$ and the angle $Q = 46^\circ$, calculate the angles of the $\triangle ABC$.

[Ans. : $\angle A = 63^\circ$, $\angle B = 67^\circ$, $\angle C = 50^\circ$]

GROUP B (Co-ordinate Geom.)

6. Show that the points $A(3, 3)$, $B(-3, -5)$ and $C(-5, -3)$ are the vertices of an isosceles triangle and find the length of its base. [Ans. : base $=4\sqrt{2}$ units of length]

7. The sides BC, CA, AB of a $\triangle ABC$ have respectively the equations (*i*) $2x+y+1=0$, (*ii*) $2x+3y+1=0$, (*iii*) $3x+4y+3=0$. Find the equation of the altitude through A.

[Ans. $x-2y+11=0$]

GROUP C—Solid Geom. & Mensuration

10. Same as Q. 7(*d*) Second part of H. S., 1961.

Or, Same as Q. 7(*a*) of H. S. (Compl.), 1960.

11. Find (*a*) the slant surface, (*b*) the volume of a right pyramid 15 cm. high, standing on a square base whose side is 16 cm. [Ans. : slant surface $=544$ sq.cm., volume $=1280$ c.c.]

Or, A zinc cistern (open at the top) measures externally 3 ft. 3 in. long, 2 ft. 3 in. broad and 2 ft. 1 in. deep and its capacity is 75 gallons. If the bottom of the cistern is 1 in. thick, find the thickness of the sides. [1 cu. ft. $=6\frac{1}{4}$ gallons]

[Ans. : $1\frac{1}{2}$ in.]

H. S. EXAMINATION, 1965

First Paper—GROUP A—Algebra

2. (*a*) Solve : $\left.\begin{array}{l}\dfrac{x}{y}+\dfrac{y}{x}=2\frac{1}{2}\\ x+y=6\end{array}\right\}$ $\left[\text{Ans. : }\begin{array}{l}x=4\\y=2\end{array}\right\}\text{ or }\left.\begin{array}{l}x=2\\y=4\end{array}\right\}\right]$

(*b*) If $(x+y)$ varies as $(x-y)$, show that (x^3+y^3) varies as (x^3-y^3).

3. (*a*) A sets out from a place and travels at the rate of 5 miles an hour. B sets out $4\frac{1}{2}$ hours after A and travels in the same direction, 3 miles the first hour, $3\frac{1}{2}$ miles the second hour, 4 miles the third hour and so on. Find in how many hours B will overtake A. [Ans. : 15 hrs.]

(*b*) If 1, w, w^2 are the three cube roots of unity, prove that $(3+3w+5w^2)^6=(3+5w+5w^2)^6=64$.

7. (*a*) Calculate the numerical value of the expression $\log\left\{\frac{(10\cdot8)^{\frac{1}{2}}\times(\cdot24)^{\frac{5}{3}}}{(90)^{-2}}\right\}$, given log 2 = ·3010300 and log 3 = ·4771213.

[Ans. : 3·3922160]

GROUP B—Trigonometry

8. (*a*) Same as Q. 7(*a*) Second part H. S., 1961

(*b*) Same as Q. 7 of H. S., 1960 (Compl.)

9. (*a*) Find the values of θ between 0° and 180° which satisfy the equation $\sin\theta+\sin5\theta=\sin3\theta$. [Ans. : 30°, 60°, 120°, 150°]

(*b*) If $A+B+C=\frac{\pi}{2}$, prove that

$$\tan A\tan B+\tan B\tan C+\tan C\tan A=1.$$

Second Paper, 1965

GROUP A—Plane Geometry

1. Draw a transverse common tangent to two given circles.

Or, Construct an equilateral triangle circumscribing a given circle.

[*Traces of construction only are required in either of the two cases.*]

3. Same as Q. 2(*a*) of H. S., 1960 (Compl.)

Show that each rectangle is equal to the square on the tangent from the point of intersection.

GROUP B—Co-ordinate Geometry

5. Show that the points $(2a, 4a)$, $(2a, 6a)$ and $(2a+\sqrt{3}a, 5a$ are the vertices of an equilateral triangle whose sides are each o length $2a$. Calculate its area. [Ans. : area = $\sqrt{3}a^2$ sq. units]

7. Express in the form $\frac{x}{a}+\frac{y}{b}=1$, the equation of the st. lin passing through the point (3, 2) and the intersection of the lin $3x+y-5=0$ and $x+5y+3=0$. $\left[\text{Ans. : }\frac{x}{\frac{7}{3}}+\frac{y}{-7}=1\right]$

Find the area of the triangle cut off from the co-ordinate axe by this line. [Ans. : $8\frac{1}{6}$ sq. units

GROUP C—**Solid Geom. & Mensuration**

10. (*a*) How do you measure the angle which a straight line makes with a given plane ?

(*b*) When is a st. line said to be perpendicular to a given plane ? (*c*) When are two planes perpendicular to one another ?

(*d*) What are skew lines ?

Illustrate your answer by suitable diagrams.

Or, Same as Q. 7(*a*) of H. S., 1960 (Compl.)

11. The base of a right prism is a triangle whose sides are 1′9″, 1′8″ and 1′1″ long. If the height is 8′, find the whole surface and volume of the prism.

[Ans. : whole surface $=37\frac{3}{4}$ sq. ft., volume $=7$ cu. ft.]

Or, Find, to the nearest tenth of a metre, the height of a conical tent which stands on a circular base of diameter 8·0 metres and which contains 90·478 cubic metres of air [$\pi=3{\cdot}1416$].

[Ans. : 5·4 metres.]

H. S. EXAMINATION, 1966

First Paper—GROUP A—**Algebra**

1. (*a*) Prove that the modulus of the product of two complex quantities is equal to the product of their moduli.

(*b*) If $\sqrt[3]{x+iy}=a+ib$, where x, y, a, b, are real and $i=\sqrt{-1}$, show that $\frac{x}{a}+\frac{y}{b}=4(a^2-b^2)$.

3. (*a*) Find the sum of $1^2+3^2+5^2+\ldots+(2n-1)^2$.

[Ans : $\frac{1}{3}n(4n^2-1)$]

(*b*) The distance through which a heavy body falls from rest varies as the square of the time it falls. A body falls from rest 64 ft. in 2 seconds, how far does it fall in 7 seconds ?

[Ans. : 784 ft.]

7. (*a*) Simplify : $\log 2+16\log\frac{16}{15}+12\log\frac{25}{24}+7\log\frac{81}{80}$, the base of the logarithm being 10. [Ans. : 1]

GROUP B—Trigonometry

8. If A, B, $A+B$ are positive acute angles, prove geometrically that $\sin(A+B)=\sin A\cos B+\cos A\sin B$.

(*b*) Prove that

$$\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan B\tan C-\tan C\tan A-\tan A\tan B}.$$

9. (*a*) $\sin A=m\sin B$, prove that $\tan\frac{A-B}{2}=\frac{m-1}{m+1}\tan\frac{A+B}{2}$.

10. (*b*) Prove that $\sin 18^\circ=\frac{\sqrt{5}-1}{4}$.

Second Paper, 1966

GROUP A—Plane Geometry

1. Draw a square equal in area to a given triangle,

Or, Draw a regular hexagon circumscribing a given circle,

(*Only traces of construction are required in each case*)

2. First part :—Same as Q. 1(*a*) of H. S., 1962.

The medians of a triangle ABC meet at O. Prove that

$$AB^2+BC^2+CA^2=3(AO^2+BO^2+CO^2),$$

3. Same as Q. 2(*a*) of H. S., 1960.

If A, B, C are three points on the circumference of a circle such that the chord AB is equal to the chord AC, prove that the tangent at A bisects the exterior angle between AB and AC.

4. Same as Q. 1(*c*) of H. S., 1961.

The tangent at the vertex A to the circumcircle of the $\triangle ABC$ meets the side BC in the point T. Prove that $\frac{TB}{TC}=\frac{AB^2}{AC^2}$.

GROUP B—Co-ordinate Geometry

5. The points $A(4, -1)$, $B(3, 2)$, $C(-1, -2)$ are the vertices of a triangle. Find the length of BC and of the altitude AD. Hence calculate the area of the triangle. Find this area also by co-ordinate geometry and check your result.

[Ans. : $BC=4\sqrt{2}$; $AD=2\sqrt{2}$; Area$=8$.]

GROUP C—**Solid Geom. & Mensuration**

10. Same as Q. 7(*d*) of H. S., 1961.

Or, Explain with the help of figures the possibe relations between

(*a*) a given straight line and a plane ;

(*b*) two given straight lines in space.

Show how through any point a plane can be drawn parallel to a given plane. (Proof not required)

11. If a solid homogeneous iron ball of 4 inches diameter weigh 9 lbs., what is the weight of an iron shell, made of same material, whose external and internal diameters are 3 inches and 6 inches respectively ? [Ans. : $72\frac{9}{64}$ lbs.]

Or, A sphere has the same volume as a right circular cone with its height equal to twice the radius of its base. Find the ratio of the curved surface of the cone to the surface of the sphere.

[Ans : $5^{\frac{1}{2}} : (16)^{\frac{1}{3}}$,]

H. S. EXAMINATION, 1967

First Paper—GROUP A—**Algebra**

2, (*a*) Solve the equations :

$$\left.\begin{array}{l} 3x+4y=18 \\ \frac{1}{x}+\frac{1}{y}=\frac{5}{6} \end{array}\right\} \qquad \left[\text{Ans.} \quad \left.\begin{array}{l} x=2 \\ y=3 \end{array}\right\} \text{ or } \left.\begin{array}{l} x=\frac{18}{5} \\ y=\frac{9}{5} \end{array}\right\} \right]$$

3. (*a*) The volume of a pyramid varies jointly as its height and the area of its base ; and when the area of the base is 60 sq. feet and the height 14 feet, the volume is 280 cubic feet. Find, by variation, the area of the base of the pyramid whose volume is 390 cu. ft. and whose height is 26 feet. [Ans : 45 sq. ft.]

(*b*) Find the sum to n terms of the series

$(3^3-2^3)+(5^3-4^3)+(7^3-6^3)+\ldots$ [Ans : $n(4n^2+9n+6)$]

7. (*a*) Prove that $\log_a m = \log_b m \times \log_a b$.

(*b*) Calculate the value of

$$\left\{\frac{(\cdot 32)^8 \times (625)^4}{(\cdot 00432)^2 \times (\cdot 3125)^3 \times 25}\right\}^{\frac{1}{5}}$$ given $\log 2 = \cdot 3010300$,

$\log 3 = \cdot 4771213$ and $\log 259569 = 5\cdot 4142524$, (correct to 7 places of decimals). [Ans : 259·569]

Group B—Trigonometry

8. (*a*) Same as Q. 10 (*a*) of H. S., 1960.

(*b*) Find the value of $\cos^2\theta + \cos^2(120^\circ - \theta) + \cos^2(120^\circ + \theta)$. [Ans : $\frac{3}{2}$]

9. (*b*) If $A + B + C = 180^\circ$, prove that

$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2 + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

10. (*a*) If $\tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\phi}{2}$, prove that

$\cos\phi = \frac{\cos\theta - e}{1 - e\cos\theta}$. [See sum No. 26 of Ex. 5 of this Book]

Second Paper—1967

Group A—Plane Geometry

1. Same as Q. 1 of H. S., 1964.

Or, Same as Q. 1. (or) of H. S., 1964 (Compl.).

2. (*a*) Same as Q. 1 (*c*) of H. S., 1961.

(*b*) ABC is a triangle, L and M are points in AB and AC respectively such that LM is parallel to BC. If $AL = 5$ cms. and $LB = 4$ cms., find the value of $\frac{\text{area } ALM}{\text{area } LMCB}$. [Ans : $\frac{25}{56}$]

3. (*a*) Same as Q. 2 (*a*) of H. S., 1962.

(*b*) Same as Q. 2 (*b*) of H. S., 1960 (Compl.)

4. (*a*) If G be the centroid of the $\triangle ABC$, show that

$$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2).$$

(*b*) Same as Q. 2 (*b*) or, of H. S., 1960 (Compl.).

GROUP B—Co-ordinate Geometry

5. Show that the three points (3, 1), (5, -5) and (-1, 13) are collinear. Find the equation of the straight line on which they lie. [Ans : $3x+y=10$]

6. (*a*) Find the co-ordinates of the centroid of the triangle whose vertices are the points (x_1, y_1), (x_2, y_2) and (x_3, y_3).

(*b*) The vertices of a triangle are (2, 7), (5, 1) and (x, 3); its area is 18. What is the value of x ? [Ans : $x=10$ or -2]

GROUP C—Solid Geometry and Mensuration

10. Same as Q. 7 (*d*) Second part of H. S., 1961.

Or, (*a*) When is a st. line said to be perpendicular to a given plane ? Illustrate your answer by a suitable diagram.

(*b*) Same as Q. 12 of H. S., 1962 (Compl.).

11. A right circular cone, 20 cm. high, has its upper part cut off by a plane passing through the middle point of its axis. If the plane of section be at right angles to the axis and if the radius of the original cone be 4 cm., find the volume of the truncated cone. ($\pi=\frac{22}{7}$).

[See 'Cone', Example 11 in this Book.]

Or, Determine the volume of a pyramid whose height is $10\sqrt{7}$ ft., and which stands on a triangle of sides 16 ft., 11 ft. and 9 ft. [See Pyramid, Example 11 in this book]

C. U. PRE-UNIVERSITY EXAMINATION—1961

Group A—**Algebra**

1. (*b*) In a "Geometrical Progression", the first term is 5, the last term is 320 and the sum of the series is 635 ; find the 4th term. [Ans. 40]

2. (*b*) Use logarithmic tables to calculate the value of $2{\cdot}41 \times (1{\cdot}24)^{\frac{1}{3}} \div (0{\cdot}78)^{\frac{1}{2}}$, correct to two places of decimals.

[Ans. 2·93]

Group B—**Trigonometry**

5. (*a*) Same as Q. 8(*a*) of H. S., 1963.

(*b*) If A, B, C are the angles of a triangle, shew that

$$\sin\frac{A}{2} + \cos\frac{B-C}{2} = 2\cos\frac{B}{2}\cos\frac{C}{2}.$$

6. (*a*) Same as Q. 8(*b*) of H. S., 1965

8. (*a*) Find the value of sin 18°. [Ans. $\frac{1}{4}(\sqrt{5}-1)$]

Group C—**Co-ordinate Geometry**

9. (*a*) The co-ordinates of A, B, C are $(-1, 5)$, $(3, 1)$ and $(5, 7)$ respectively. D, E, F are the middle points of BC, CA, AB respectively. Calculate the area of the triangle DEF.

[Ans. 4 sq. units]

(*b*) Obtain the equation of the straight line through the point $(2, 1)$ and perpendicular to the line joining the points $(2, 3)$ and $(3, -1)$. [Ans. $x - 4y + 2 = 0$]

10. (*a*) Obtain the equation of the locus of a point which moves in the plane of (xy) in such a way that its distance from the point $(2, 3)$ is always two-thirds of its distance from y-axis.

[Ans. $5x^2 + 9y^2 - 36x - 54y + 117 = 0$]

12. (*a*) Obtain the equations of the lines which bisect the angles between the lines (*i*) $a_1x + b_1y + c_1 = 3$ and

(*ii*) $a_2x + b_2y + c_2 = 0$.

Group D—**Solid Geometry & Mensuration**

13. Same as Q. 7(*b*) Second Part of H. S., 1960.

14. (*a*) Find the locus of a point in space equidistant from two given points.

(*b*) Three solid spheres of gold whose radii are 1 cm., 6 cms., and 8 cms. respectively are melted into a single gold sphere. Find the radius of the sphere so formed. [Ans. 9 cm.]

C. U. PRE-UNIVERSITY EXAM.—1962

Group A—**Algebra**

1. (*a*) If $x+iy=\dfrac{\sqrt{3}-i\sqrt{2}}{2\sqrt{3}-i\sqrt{2}}$, where $i=\sqrt{-1}$, find x and y.

$$\left[\text{Ans. } x=\frac{4}{7},\ y=-\frac{\sqrt{6}}{14}\right]$$

(*b*) In an *A. P.* the first term is 2, the last term 29, the sum 155 ; find the common difference. [Ans. 3]

2. (*a*) If P varies as the sum of two quantities of which one varies directly as x and the other inversely as x and if $P=6$ when $x=4$ and $P=3\frac{1}{3}$ when $x=3$, find P when $x=2$. [Ans. 0]

(*b*) Use logarithmic tables to calculate the value of

$$\frac{(48{\cdot}7)^{\frac{1}{3}}\times({\cdot}00321)^{\frac{1}{2}}}{0{\cdot}372}. \qquad [\text{Ans. } {\cdot}5563]$$

Group B—**Trigonometry**

5. (*a*) Shew, geometrically, that if A and B are positive acute angles and $(A+B)<90^\circ$, then

$$\sin(A+B)=\sin A\cos B+\cos A\sin B.$$

(*b*) Show that $\cos A+\cos B+\cos C+\cos(A+B+C)$

$$=4\cos\frac{A+B}{2}\cos\frac{B+C}{2}\cos\frac{C+A}{2}.$$

Group C—**Co-ordinate Geometry**

9. (*a*) Find the angle between the two st. lines

$$y=mx+c,\ y=m'x+c'.$$

(*b*) Obtain the equations to the st. lines each of which passes through the point $(2,\ -1)$ and intersects the axes of co-ordinates at points equidistant from the origin and calculate the angle between them. [Ans. $x+y=1,\ x-y=3$; a right angle.]

12. (*b*) A point P moves in the plane of (xy) in such a way that its distances from the lines $12x+5y-4=0$ and $3x+4y+7=0$ are equal ; obtain the equation to the locus traced out by P.

[Ans. $7x-9y-37=0$, or, $99x+77y+71=0$]

Group—D—**Solid Geometry & Mensuration**

13. (*a*) Same as Q, 7 (*a*) of H. S., 1960 (Compl.)

(*b*) The diagonal of a rectangular block is 10 cms., and the sum of the lengths of its edges is 80 cms., calculate the total area of the outer surface of the block. [Ans. 6300 sq. cm.]

14. (*a*) Shew that if a st. line is perpendicular to a plane, then every plane passing through it is perpendicular to that plane.

14. (*b*) A right circular cone is 10 cms. high and its slant height is 15 cms. Calculate the volume of the cone. $[\pi=\frac{22}{7}]$.

[Ans. $1309\frac{11}{21}$ cu. cms.]

C. U. PRE-UNIVERSITY EXAM.—1963

Group—A—**Algebra**

1. (*a*) Express $(3+i)(4+3i)(5+7i)$ in the from of $A+iB$, where $i=\sqrt{-1}$. [Ans. $-46+128i$ where $-46=A,\ 128=B$]

(*b*) If $x,\ y,\ z$ be the p^{th}, q^{th} and r^{th} terms respectively of a G. P., prove that $x^{q-r}\,y^{r-p}\,z^{p-q}=1$.

2. (*a*) Solve $x+y=5,\ x^2+2y^2=17$.

[Ans. $x=3,\ y=2$, or, $x=\frac{11}{3},\ y=\frac{4}{3}$]

4. (*b*) Prove that $7\log\frac{16}{15}+5\log\frac{25}{24}+3\log\frac{81}{80}=\log 2$.

Group B—Trigonometry

5. (*a*) Prove geometrically that if A and B are positive acute angles and $A+B<90^\circ$, $\cos(A+B)=\cos A \cos B-\sin A \sin B$.

(*b*) Express $\cos 4\theta$ in terms of $\cos\theta$.

[Ans. $8\cos^4\theta-8\cos^2\theta+1$]

6. (*a*) Show that $\tan 3\theta=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$.

Droup C—Co-ordinate Geometry

9. (*a*) Same as Q. 5 (*a*) of H. S., 1963 (Compl.)

(*b*) If (x_1, y_1), (x_2, y_2) and (x_3, y_3) be the co-ordinates of the vertices A, B, C respectively of the triangle ABC and if D is the middle point of BC, find the co-ordinates of G which divides AD such that $AG=2GD$.

[Ans. $\{\frac{1}{3}(x_1+x_2+x_3), \frac{1}{3}(y_1+y_2+y_3)\}$]

10. (*a*) Find the equation of the st. line passing through the point (3, 5) and parallel to the line $4x-3y+1=0$.

[Ans. $4x-3y+3=0$]

11. (*a*) Find the locus of the point whose distance from the point $(-1, 1)$ is equal to its perpendicular distance from the st. line $x+y+1=0$. [Ans. $x^2+y^2-2xy+2x-6y+3=0$]

Group D—Solid Geometry & Mensuration

13. (*a*) Prove that two intersecting planes cut one another in a straight line and in no point outside it.

(*b*) Same as Q. 7 (*a*) Second Part of H. S., 1960.

14. (*a*) Three solid gold spherical beads of radii 3, 4, 5 cms. respectively are melted into one solid spherical bead. Find its radius.

[Ans. 6 cms.]

(*b*) The volume of a right circular cylinder and a right circular cone standing on the same base are as 3 : 2. Show that the height of the cone is double the height of the cylinder.

C. U. PRE-UNIVERSITY EXAMN.—1964

1. (*b*) Express $(\sqrt{3}-\sqrt{2}i)(2\sqrt{3}-\sqrt{2}i)$ in the form of $A+iB$, where $i=\sqrt{-1}$ and A, B are real.

[Ans. $4-3\sqrt{6}i$ where $A=4$, $B=-3\sqrt{6}$]

2. (*b*) Solve $\left.\begin{array}{l} x^2+y^2=25 \\ x\ +y\ =\ 7 \end{array}\right\}$ $\left[\text{Ans. } \left.\begin{array}{l} x=3 \\ y=4 \end{array}\right\} \text{ or } \left.\begin{array}{l} x=4 \\ y=3 \end{array}\right\}\right]$

4. (*a*) Simplify : $16\log\frac{16}{15}+12\log\frac{25}{24}+7\log\frac{81}{80}+\log 2$, the base of the logarithm being 10. [Ans. 1]

(*b*) Evaluate (using logarithmic tables) :

$$\frac{5{\cdot}631\times42{\cdot}13\times{\cdot}2783}{2{\cdot}451\times{\cdot}8392\times12{\cdot}61}.$$

[Ans. 2·545]

5. (*a*) Same as Q. 9 (a) of H. S., 1961 (Compl.).

(*b*) Same as Q. 10(*b*) of H. S. 1961 (Compl.).

8. (*b*) From an aeroplane vertically above a st. horizontal road, the angles of depression of two consecutive mile-stones are observed to be 30° and 45°. If the mile-stones be on the opposite sides of the aeroplane, show that its height, in miles, above the road is $\frac{1}{1+\sqrt{3}}$.

9. (*a*) The three points, whose co-ordinates are (x_1, y_1), (x_2, y_2) and (x_3, y_3) lie in a st. line. Show that

$$x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0.$$

(*b*) Find the angle between the straight lines

$x-\sqrt{3}y=1$ and $\sqrt{3}x-y=4$. [Ans. 30° or 150°]

13. (*a*) Same as Q. 7 (a) of H. S. 1960 (Compl.)

(*b*) If a triangle revolves about its base, prove that the locus of its vertex is a circle.

14. (*a*) A right prism stands on a triangular base whose sides are 18″, 20″, 34″. If the height of the prism is 10″, find the area of its total surface. [Ans. 1008 sq. in.]

(*b*) How many square feet of canvas are required for a conical tent 24 feet high, the diameter of the base being 14 feet?

$[\pi=\frac{22}{7}]$ [Ans. 550 sq. ft.]

B. U. ENTRANCE EXAMINATION—1961

Group A—Algebra

1. (*b*) Solve $x+y+3\sqrt{x+y}=x^2+y^2=10$.
[Ans - $x=3$, $y=1$; or, $x=1$, $y=3$]

(ii) $\frac{x^2}{y}+\frac{y^2}{x}=\frac{9}{2}$, $\frac{1}{x+y}=\frac{1}{3}$ [Ans : $x=2$, $y=1$; or, $x=1$, $y=2$]

4. (*a*) Prove that $\log_b a \times \log_c b \times \log_a c = \log_a a$.

(*b*) Find S_n, when $S_n=1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}$

$\left[\text{Ans}: \ 2-\frac{1}{2^{n-1}}\right]$

Calculate the lowest value of n so that $2-S_n<\frac{1}{100}$. [Ans : 8]

Group B—Trigonometry

5. (*a*) Prove geometrically
$\cos(A+B)=\cos A\cos B-\sin A\sin B$,

when $0<A<\frac{\pi}{2}$, $0<B<\frac{\pi}{2}$ and $A+B<\frac{\pi}{2}$.

(*b*) If $A+B+C=\pi$, prove that
$\cot A\cot B+\cot B\cot C+\cot C\cot A=1$.

Group C—Co-ordinate Geometry

8. (*a*) Find the equation to the straight line passing through the intersection of lines $x+2y-3=0$ and $3x+4y+7=0$ and perendicular to the straight line $y-x=8$. [Ans : $x+y+5=0$]

(*b*) Show that three points (4, 2), (7, 5) and (9, 7) are in one straight line and find the equation of the line of collinearity.
[Ans : $x-y=2$]

Group D—Solid Geometry & Mensuration

12. (*a*) Same as Q. 7(*a*) of H. S., 1960 (Compl).

(*b*) Same as Q. 13 of H. S., 1962

13. (*a*) Same as Q. 14 (*a*) of C. Pre-U., 1962.

B. U. ENTRANCE EXAM.—1962

Group A—**Algebra**

1. (*a*) Represent geometrically the numbers $3+2i$, $6+4i$ and $9+6i$. Find their moduli.

[Ans : Moduli $=+\sqrt{13}, +2\sqrt{13}, +3\sqrt{13}$]

(*b*) Solve : (i) $x^2+y^2+xy=84$, $x+y=10$

[Ans : $x=8$, $y=2$; or $x=2$, $y=8$]

(ii) $y^x=4$, $y^2=2^x$.

[Ans : $x=2$, $y=\pm 2$; or, $x=-2$, $y=\pm\frac{1}{2}$]

4. (*b*) If $\dfrac{\log x}{b-c}=\dfrac{\log y}{c-a}=\dfrac{\log z}{a-b}$, prove that $x^a y^b z^c=1$.

Group B—**Trigonometry**

5. (*a*) Same as Q. 9 (*a*) of H. S. 1960 (Compl.)

(*b*) If $A+B+C=\pi$, prove that

$$\cos^2 A+\cos^2 B+2\cos A\cos B\cos C=\sin^2 C.$$

8. (*a*) Find out the value of $\sin 18°$.

Group C—**Co-ordinate Geometry**

9. (*a*) Prove analytically that the line joining the middle points of any two sides of a triangle is half the third side.

(*b*) Find the equations of the medians of the triangle whose vertices are the points $(3, 2)$, $(1, -1)$, $(-19, -9)$ and show that they are concurrent.

[Ans. $7x-12y+3=0$, $5x-18y-23=0$, $19x-42y-17=0$]

Group D—**Solid Geometry & Mensuration**

13. (*a*) Same as Q. 7 (d) Second part of H. S. 1961

(*b*) Show that all st. lines drawn perpendicular from a given point to a system of parallel st. lines in space are coplanar.

14. (*a*) How many solid circular cylinders of length 8 in. and diameter 6 in. can be made out of the material of a solid sphere of radius 6 inches ? [Ans. 4]

(*b*) A right circular cone 15 cm. high, the radius of the base being 8 cm., has its upper part cut off by a plane through the middle point of its axis and parallel to the base. Find the volume of the truncated cone. [Ans : 880 cu. cm.]

B. U. ENTRANCE EXAM.—1963

Group A—**Algebra**

1. (*a*) Solve (any two) :— (i) $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{5}{2}$, $x+y=5$.

(ii) $x^{\frac{1}{3}}+y^{\frac{1}{3}}=3, x+y=9$ (iii) $x^y=y^2$, $y^{2y}=x^4$.

[Ans : (i) $x=1, y=4$, or, $x=4, y=1$: (ii) $x=1, y=8$ or, $x=8, y=1$; (iii) $x=\pm 2, y=2$, or, $x=\pm\frac{1}{2}, y=-2$.]

(*b*) Three globes of gold whose radii are 3, 4 and 5 inches are melted and formed into a new globe ; find the radius of the new globe, given that the volume of a globe varies as the cube of its radius. [Ans. 6 inches]

Group B—**Trigonometry**

6. (*a*) Same as Q. 5(*a*) of C. Pre-U., 1963.

(*b*) If $\sin\theta+\sin\phi=a$, $\cos\theta+\cos\phi=b$,

prove that $\tan\frac{\theta-\phi}{2}=\pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$.

7. (*a*) If $\alpha+\beta=\gamma$, show that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma$

$$=1+2\cos\alpha\cos\beta\cos\gamma.$$

Group C—**Co-ordinate Geometry**

10. (*a*) Show that the area of the triangle, the co-ordinates of whose angular points are $\left(a, \frac{1}{a}\right)$, $\left(b, \frac{1}{b}\right)$ and $\left(c, \frac{1}{c}\right)$ is

$$\frac{(b-c)(c-a)(a-b)}{2abc}.$$

(*b*) If the centroid of a triangle is (1, 4) and two of its vertices are (4, −3) and (−9, 7), find the other vertex. [Ans. (8, 8)]

11. (*a*) Show that the equation

$a_1x+b_1y+c_1+\lambda(a_2x+b_2y+c_2)=0$, where λ is an arbitrary constant, represents any straight line passing through the point of intersection of the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$.

(*b*) Find the equation of the straight line which passes through (x_1, y_1) and is perpendicular to the join of (x_2, y_2) and (x_3, y_3). [Ans. $(y_3-y_2)(y-y_1)+(x_3-x_2)(x-x_1)=0$.]

GROUP D—Solid Geometry & Mensuration

14. (*a*) If two intersecting planes are each perpendicular to a third plane, prove that their line of intersection is also perpendicular to the plane.

(*b*) If of three lines of intersection of three planes, two be parallel, show that the third will also be parallel to the other two.

15. (*a*) The length, breadth and the height of a closed box are 12 in. 10 in., and 8 in. respectively and the total inner surface is 376 sq. in. If the walls of the box are uniformly thick, find the thickness. [Ans. 1 inch.]

(*b*) If S be the area of the curved surface, v the volume, h the height and α the semi-vertical angle of a right circular cone, prove that $S=\dfrac{\pi h^2 \sin\alpha}{\cos^2\alpha}$ and $v=\frac{1}{3}\pi h^3 \tan^2\alpha$.

B. U. ENTRANCE EXAM.,—1964

1. (*a*) Same as Q. 3 (*a*) H. S., 1963 (Compl.)

(*b*) Show that a real value of x will satisfy the equation $\dfrac{1-ix}{1+ix}=a-ib$, if $a^2+b^2=1$, the constants a, b being real.

2. (*a*) Solve (any two) :

(i) $\dfrac{x^2}{y}+\dfrac{y^2}{x}=18,\ x+y=12.$

[Ans : $x=4, y=8$, or, $x=8, y=4$]

(ii) $\dfrac{2x+\sqrt{x}}{2x-\sqrt{x}}+6.\dfrac{2x-\sqrt{x}}{2x+\sqrt{x}}=5.$ [Ans : $x=\frac{9}{4}$ or 1]

(iii) $8^x.4^y=128,$
$9^{x+y}=27^{xy}$ $\left[\text{Ans : } \left.\begin{matrix} x=1 \\ y=2 \end{matrix}\right\} \text{ or } \begin{matrix} x=\frac{14}{9} \\ y=\frac{7}{6} \end{matrix}\right]$

(*b*) If $x+y \propto z$ when y is constant and $x+z \propto y$ when z is constant, then show that $x+y+z \propto yz$ when both y and z vary.

6. (*a*) Same as Q. 8 (*a*) of H. S. 1965.

(*b*) Prove that $4(\cos^3 10^\circ+\sin^3 20^\circ)=3(\cos 10^\circ+\sin 20^\circ)$.

9.(*b*) A vertical post 15 ft. high is broken at a certain height and its upper part, not completely separated, meets the ground at an angle of 30°. Find the height at which the post is broken.

[Ans : 5 ft.]

10. (*a*) A point P divides the st. line AB internally in the ratio $\alpha : \beta$. The co-ordinates of A and B are (a_1, a_2) and (b_1, b_2) respectively. Find the co-ordinates of P.

$$\left[\text{Ans} : \left(\frac{a_1\beta+b_1\alpha}{\alpha+\beta}, \frac{a_2\beta+b_2\alpha}{\alpha+\beta}\right)\right]$$

(*b*) Find the equation of the locus of a point $P(x, y)$ if the ratio of its distance from the point $(-10, 0)$ to its distance from the point $(1, 0)$ be $2 : 1$. [Ans : $3x^2+3y^2-28x-96=0$]

11. (*a*) Find an expression for the angle between the two st. lines given by $Ax+By=0$ and $A'x+B'y=0$ respectively.

[See Text Book]

(*b*) Find the distance of the point of intersection of the two st. lines $2x-3y+17=0$ and $3x+4y=0$ from the st. line $4x-3y=0$. [Ans : -5 (numerically $=5$ units)]

14. (*a*) What do you mean by (i) the angle between a plane and a straight line ; (ii) the angle between two planes ?

If a right angle revolves about one of its arms, then prove that the other arm describes a plane.

(*b*) The base of a right prism is a triangle whose perimeter is 15 cm. and the radius of the in-circle of the triangle is 3 cm. If the volume of the prism be 270 c.c., find its height.

[Ans : 12 cm.]

15. (*a*) Same as Q. 7(*b*) Second part of H. S. 1960.

(*b*) Same as Q. 13 (*b*) of B. U. E., 1962.

CORRIGENDUM

Algebra :

Page 53, in line 20, read $\left\{\frac{n(n+1)}{2}\right\}^2$ for $\left\{\frac{n(n+1)}{4}\right\}^2$.

,, 63, the sum 12 is $9+7+5+\cdots+(-25)$.

,, 69, in Ex. 9, read $r=\frac{3}{9}$ for $r=\frac{9}{3}$.

,, 97, in Ex. 1, read $\because\ x \propto y$ for $\because\ x \propto b$.

Trigonometry :

Page 86, in the last line of Ex. 20, read $3x-x^3$ for $3x-x^2$.

Geometry and Co-ordinate Geometry :

Page 129, in line 13, read $\mathrm{AB} \perp \mathrm{BE}$ for $\mathrm{AB} \perp \mathrm{BP}$.

,, 147, in line 24, read $\frac{\mathrm{RQ}+\mathrm{PR}}{\mathrm{PR}}$ for $\frac{\mathrm{PQ}+\mathrm{PR}}{\mathrm{PR}}$.

,, 150, the diagram should be inverted.

,, 160, in line 13, read $+\left(\frac{y_1+y_2}{2}-\frac{y_1+y_3}{2}\right)^2$.

,, 164, in sum 21, read $(-2, 3)$ for $(-3, 3)$.

,, 203, in line 7, read $-\frac{c_2}{b_2}$ for $-\frac{c_2}{b^2}$;

and in line 10, read $\times\frac{a_2}{b_2}$ for $\times\frac{a_2}{b^2}$.

,, 212, in last but one line, read $4y$ for $4x$.

Appendix :

Page 20, in last line of Ex. 13. read 19 in. for 9 in.

Questions :

Page xxiii, read Q. 8(*a*) and Q. 8(*b*) same as Q. 5(*a*) and Q. 6(*a*) of C. Pre-U. 1961.

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						5	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	16	20	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	18	22	26	29	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	6	10	13	16	19	23	26	29
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	14	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
16	2041	2068	2095	2122	2148						3	6	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	3	5	8	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	17	19	21
						2672	2695	2718	2742	2765	2	4	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	123	456	789
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0 0 1	1 1 1	2 2 2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0 0 1	1 1 1	2 2 2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0 0 1	1 1 1	2 2 2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0 0 1	1 1 1	2 2 2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0 1 1	1 1 2	2 2 2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0 1 1	1 1 2	2 2 2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0 1 1	1 1 2	2 2 2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0 1 1	1 1 2	2 2 2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0 1 1	1 1 2	2 2 3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0 1 1	1 1 2	2 2 3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0 1 1	1 1 2	2 2 3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0 1 1	1 2 2	2 2 3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0 1 1	1 2 2	2 2 3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0 1 1	1 2 2	2 3 3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0 1 1	1 2 2	2 3 3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0 1 1	1 2 2	2 3 3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0 1 1	1 2 2	2 3 3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0 1 1	1 2 2	2 3 3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0 1 1	1 2 2	2 3 3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0 1 1	1 2 2	3 3 3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0 1 1	1 2 2	3 3 3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0 1 1	2 2 2	3 3 3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0 1 1	2 2 2	3 3 3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0 1 1	2 2 2	3 3 4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0 1 1	2 2 2	3 3 4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0 1 1	2 2 2	3 3 4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0 1 1	2 2 3	3 3 4
27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0 1 1	2 2 3	3 3 4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0 1 1	2 2 3	3 4 4
29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0 1 1	2 2 3	3 4 4
30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0 1 1	2 2 3	3 4 4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0 1 1	2 2 3	3 4 4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0 1 1	2 2 3	3 4 4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0 1 1	2 2 3	3 4 4
34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1 1 2	2 3 3	4 4 5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1 1 2	2 3 3	4 4 5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1 1 2	2 3 3	4 4 5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1 1 2	2 3 3	4 4 5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1 1 2	2 3 3	4 4 5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1 1 2	2 3 3	4 5 5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1 1 2	2 3 4	4 5 5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1 1 2	2 3 4	4 5 5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1 1 2	2 3 4	4 5 6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1 1 2	3 3 4	4 5 6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1 1 2	3 3 4	4 5 6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1 1 2	3 3 4	5 5 6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1 1 2	3 3 4	5 5 6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1 1 2	3 3 4	5 5 6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1 1 2	3 4 4	5 6 6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1 1 2	3 4 4	5 6 6

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

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